Index

Aspetti generali

1. The ArAl Project

The ArAl Project is meant to innovate the teaching of arithmetic and algebra in both primary and lower secondary school. The project is located within the *early algebra* theoretical framework, according to which *the main cognitive obstacles in learning algebra often arise in arithmetical contexts, in unexpected ways, and may later bring about conceptual obstacles – that may be insurmountable- to the development of algebraic thinking*.

A brief illustration of the main points of this hypothesis is needed here.

The international literature dealing with research on mathematics learning and in particular on the learning of algebra and on related difficulties – at different age levels, from the beginning up to university- highlights a widespread crisis of the traditional teaching of algebra. The identified reasons are very different in nature: *cognitive* reasons (algebra is difficult *per se*), *psychological* reasons (algebra intimidates), *social* reasons (the environment passes on *phobic* attitudes towards mathematics), *pedagogical* reasons (students seem to be less and less motivated towards studying especially when higher performances are requested), *didactical* reasons (stereotyped and inadequate methods).

Algebra, as language characterising a *higher* mathematics, represents a sort of wall for many students, mainly because they often have a weak conceptual control of *meanings* of both algebraic *objects* and *processes*. In the last twenty years research focused on a wide number of possible approaches to develop this type of control, for instance problem solving, functional approach, approach to generalisation.

Among other the *linguistic* approach is becoming increasingly important: it starts from a conception of *algebra as a language*. In this perspective the strong hypothesis of ArAl Project is that there is an *analogy* between ways of learning natural language and ways of learning algebraic language; the *babbling* metaphor can be useful to clarify this point of view.

Learning a language the child gradually appropriates its meanings and rules, developing them through imitation and adjustments up to school age when he will learn to read and reflect on *grammatical* and *syntactical* aspects of language. In the traditional teaching and learning of algebraic language the study of rules is generally privileged, as if formal manipulation could precede the understanding of meanings. The general tendency is to teach the syntax of algebra and leave its semantics behind. Mental models characterising algebraic thinking should rather be constructed within an arithmetical environment – starting from early years of primary school – through initial forms of *algebraic babbling*, teaching the child *how to think arithmetic algebraically*. In other words, algebraic thinking should be progressively constructed in the child as both an instrument and an object of thinking, strictly interweaved with arithmetic, starting from its meanings.
For this purpose it is necessary to construct an environment able to stimulate an autonomous elaboration of algebraic babbling and consequently to favour the experimental appropriation of a new language in which rules may be gradually located, within the constraints of a didactical contract that tolerates initial moments of syntactical 'promiscuity'.

Aspetti generali

2. ArAl Units

The Units are an important result of ArAl Project and they are designed for a wide diffusion of the project itself; they can be viewed as models of processes of arithmetic's teaching in an algebraic perspective and are meant to provide teachers an opportunity to reflect on both their knowledge and their modus operandi in their classes before offering teaching paths to implement in class.

The 'fine tuning' of each Unit of ArAl project is the result of a process lasting at least three years, organised through a sequence of phases:

a) The choice of themes to be investigated

- At the beginning of each school year the themes around which experimental projects will be articulated are elaborated;

b) Experimental setting in the classes: joint lessons, minutes

- each project – launched by an extremely flexible sequence of problem situations- is developed throughout the year in several experimental classes; in primary school classes, teachers and teachers-researchers[1] simultaneously carry out the project through joint lessons;
- class teachers <u>write minutes</u> for every meeting (taking notes, making audio recordings or vide recordings in different situations) collecting a high amount of documental material (discussions, written <u>protocols</u>, methodological notes, unforeseen events, reflections, hints and so on);
- class minutes – which represent a fundamental instrument for the analysis of the teaching/learning process within the project- are transcribed into electronic form by class teachers and sent out to teachers-researchers who carried out the activities in a joint lesson;
- purposefully collected minutes are periodically spread to the group;
- in between two subsequent joint lessons class teachers clarify and deepen with their students some aspects which were left incomplete, propose reinforcing problems, collect meaningful materials.

c) Transition to the Units

- at the end of the school year minutes of each class are globally revisited on the basis of carried out discussions and organised in the form of embryo of a Unit to be tested later in classes participating in the project as well as in external classes.

d) Writing up the Units in their final version

- when the collected elements are considered sufficient the Unit is organised in its final version through the elaboration of the most significant parts of collected minutes (which may be over one hundred in the case of

[1] The role of teachers-researchers is an Italian peculiarity. In the early 70s, following innovative ideas brought forward during the 60s, spontaneous meetings between university and non-university teachers give rise to this figure. From these meetings Research Kernels in Mathematics Education are constituted, patronised by CNR and by CIIM (Italian Committee for Mathematics Teaching), a sub-organisation within UMI (Italian Mathematical Union).

Aspetti generali

compelling Units).

- The Units are structured so that they can:
    - Describe – in the left hand side of each page- a reasoned sequence of synthesised didactical paths carried out with constructive modalities,
    - Make transparent- in the right hand side- aspects, deduced from analytical reading of minutes (see previous point b), which can help the teacher in the implementation: methodological choices, enacted dynamics, key elements in processes, extensions, pupils' potential behaviour, difficulties and so on.

e) The Unit is published.

The Units are meant to be used in the classroom but their actual implementation requires a theoretical study. Two basic instruments of the project have been elaborated to this purpose: the *reference theoretical framework* and the *Glossary*.

Four sites host ArAl materials:
(1) www.aralweb.it
This is the official web-site, documenting the project in its scientific, methodological and educational aspects, concerning materials already published in the ArAl series as well as 'in progress' materials. Increasing space is given to the use of new technologies in mathematics education – from primary to lower secondary school- in an *e–learning* perspective.
(2) www.eun.org
Educational platform of the European Community which hosts the ArAl Community.
(3) www.matematica.unimo.it/0attività/formazione/grem
English version of the project; it is within the website of the Mathematics Department of the University of Modena and Reggio Emilia.
(4) www5.indire.it:8080/set/aral/aral.htm
The web-site is inside the Indire pages. It includes part of the materials of ArAl project, selected together with other 27 within the national contest SeT [2](2001) and funded by MPI.

[2] SeT: Special project for scientific and technological innovation

Aspetti generali

3. The Glossary

Some terms that appear in each Unit constitute the **keywords** in the theoretical context of the Project. A correct understanding of these terms permits to set the proposed activities within a framework that is consistent with the inspiring principles as well as with other Units.

For this reason the Glossary can be viewed as the actual turning point for the whole ArAl project, in that it is constructed in order to promote and support, together with the Units, reflection by the teacher not only around themes developed in them, but , and more generally, on knowledge and convictions that lead him/her to explore delicate links through which the complex relationships linking arithmetic and algebra are made explicit.

The set of keywords elaborated so far is destined to be expanded: as to November 2003 it consists of 71 terms, mutually interconnected through a rich net of cross-references, and collected in the Glossary published in the first volume of this series. The terms belong to very different categories: original constructs (*algebraic babbling*, *inebriation by symbols*, *semantic persistence*); references to other theoretical constructs (*didactical contract, negotiation, pseudo-equation*); common terms used with a particular meaning (*diary, discussion, metaphor*); words belonging to the context of linguistics (*paraphrase, syntax, translating*) or to a mathematical context (*unknown, multiplicative form, equal*); adjectives that assume nuances of meaning that differ from their own (*naive, opaque, transparent*).

4. This Unit's keywords

For the reader's ease we report here all the Glossary's keywords that are referred to within the Unit; they are underlined the first time they appear.

Algebraic babbling
Arguing
Brioshi
Canonical / non canonical (representation, form)
Collective (exchange, discussion)
Describing (in mathematical language)
Diary of joint sessions activities
Didactical contract
Discussion → Collective (exchange, discussion)
Equal (sign)
Exchange → Collective (exchange, discussion)
Formal coding (writing in a formula)
Formal/formalization → translating/translation
Language (mathematics as a)
Letter (use of)
Metaphor → didactical mediator
Multiplicative (form) → Additive (form, representation)
Notation (mathematical) → Sentence (mathematical)

Aspetti generali

Opaque / Transparent (as concerns meaning) → Procedural
Paraphrase
Process / product
Protocol
Regularity
Relationship
Represent/ solve
Representation
Result → Process / product
Semantics/ syntax
Sentence (mathematical)
Sharing → Collective (exchange, discussion)
Social (achievement, construction) → Collective (exchange, discussion)
Social mediation
Solution → Represent/solve
Spot → Didactical mediator
Syntax / semantics → Semantics / syntax
Translating/translation
Transparent → Opaque / Transparent (as concerns meaning)
Verbalise, verbalisation

5. The Unit

The ArAl Project Units are characterised by a constant presence of activities that entail a search for regularities, in particular in Unit 4: *Search for regularities: the numbers grid* and in Unit 5: *Pyramids of numbers*. Activities requiring the discovery of regularities in structures are precious for the formation of algebraic thinking, since they favour transition to *generalisation*: making pupils grasp a situation of regularity means teaching them how to identify the key for an algebraic reading of the considered *structure*.

Algebra tends to unify the study of situations that are more or less similar, beyond factors like context, type of involved elements and their numerical values: in other words algebra goes beyond those elements of diversity that hinder – or even block- a process of recognition of a common basis. Similarities are recognised through the creation of correspondences among those elements of the examined situations that satisfy the relationships linking them: this process is proper of reasoning *by analogy*.

When these correspondences are built situations are said to be *analogous* or presenting the *same structure*, or else, that they are linked by a *structural analogy*. The term structure refers to the net of relationships that connect elements involved in one particular situation. Situations are said to be analogous when they share this net.

Searching for regularities can give a lot of information to teachers: they ca understand whether pupils learn to tackle problem situations with method and systematically, whether they are able to express themselves with appropriate language (also using formulae), whether they can make predictions and verify them.

Aspetti generali

6. Educational aspects

The Unit proposes a path that – through individual, group or class exploration and collective discussion on intuitions and discoveries- leads pupils to achieve an initial concept of regularity in a sequence and to the possibility of *describing* it through mathematical symbolism.

Initially concrete situations are presented: friezes, drawings, frames constructed by repeating a stencil- so that the perceptual aspect may help pupils to understand the environment in which they start carrying out their explorations.

The perceptual aspect is fundamental. The pupil must learn to *see* the sequence of drawings or objects from a 'productive' point of view and – if necessary- to modify his spontaneous point of view, which can be paralysing with respect to understanding. *Note 2: different perceptions of a frieze* deals with this aspect: on the basis of experimental activities carried out in infant schools and with teachers we hypothesised that initial difficulties in this field concern children, teen-agers and adults.

Intermediate situations in the Unit deal with necklaces made of pearls of various colours and set in different orders. Research is more refined than in previous situations and search for regularities is analysed more in depth.

The third phase, the most advanced, deals with arithmetic progressions or rather, arithmetic sequences (let us remind that an arithmetic progression is a sequence of numbers in which the difference between a term and its antecedent is constant). Increasing importance is given to a *methodical reading* of the context within a search for *variants* and *invariants* that characterise it. Processes of coding and decoding are learned and practised as a way to represent structures of sequences through algebraic language as well as to deduce a sequence from the interpretation of the mathematical sentence that represents it.

7. Terminology and symbols

Phase	Sequence of situations of increasing difficulty referring
Situation	Problem around which individual, group and class ac-
Blackboard	The frame with a black background and white signs
Expansion	The grey frame contains a working hypothesis on a

Aspetti generali

Supplementary activity	The grey frame contains an extension to topics related to those developed in previous Situations
Note	The grey frame contains either a methodological or an operative hint for the teacher
…. n	A pedicel near a term or at the end of a sentence refers to an explanation in the right hand column of Comments.
	In the grey bordered frame a problem situation is described. The proposed text is only indicative; its formulation represents the outcome of a social mediation between teacher and class.
Underlined term	An underlined term refers back to a correspondent voice in the Glossary. The term is underlined the first time it appears in the text.
Diary	The frame contains a meaningful excerpt of a discussion taken from the minutes of one of the activities carried out in a class participating in the ArAl project. Some symbols synthesise the type of intervention:
√	√ Teacher's intervention
	A pupil's intervention
	Summary of some interventions
	Result of a collective discussion (a principle, a rule, a conclusion, an observation and so on).
→	Two arrows at the end of a page and at the beginning of the next page mean that the text (Diary, protocol etc.) in which they are included is not interrupted.

Aspetti generali

8. Phases and expansions, Situations and topics

PHASES	SITUATIONS	TOPICS
First	**1 - 2**	Friezes, stencils, linguistic regularities
Second	**3 - 7**	Search for regularities on variously coloured beads' necklaces.
Third	**8 - 15**	Search for regularities in arithmetic progressions

9. Distribution of situations in relation to pupils' age

Distribution represents an indicative proposal based on the experience made in the project's classes. It is extremely important whether pupils who tackle these exploratory activities have already carried out other activities within ArAl project or not, or have dealt with themes related to an early approach to algebraic thinking. This means that they have or not acquired pre-requisites in relation with themes such as the different representations of a number, the use of letters, or general aspects like a collective reflection on mathematical objects or an approach to generalisation.

For instance, if a secondary school teacher deems it appropriate he/she can develop the first phase situations in a 6^{th} grade as moment of playful exploration. At the same time, if a primary school teacher estimates that his/her 4^{th} grade has not made sufficient experience in these fields he/she can stop at phase II or at the initial situations of phase III and restart the activity at a later moment or the subsequent year.

		PHASES AND SITUATIONS																			
		I		**II**								**III**									
		1	**2**	**3**	**E1**	**4**	**E2**	**E3**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**E4**	**E5**	**15**
pri	3																				
	4																				
	5																				
sec	1																				
	2																				
	3																				

The Unit

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	**Comments**

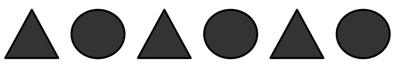
First phase

1. Friezes and stencils

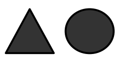
The activity is developed starting from the analysis of friezes, that pupils already know from infant school, since they have constructed them through stencils (stamps) in the form of 'patterns' or 'decorations'.
Work involves presentation of some friezes and their analysis with pupils, searching for the module that generates them, and for their generation law.

Diary 1 (3rd grade)

√ A frieze is proposed and pupils are asked how they might construct a stamp that enables the realisation of the frieze itself.



«I would use two stamps, a triangular one and a circular one »
√ «what if we wanted to use only one stamp?»
Discussion leads to the following solution:



√ «In this frame, how do we know what symbol takes the 15th place?»
(the same pupil as before) «All drawings in the even places are circles, and all those in odd places are triangles. Therefore in the 15th place there is a triangle »
Pupils immediately grasp the sense of their classmate's conclusion. Other analogous questions present no difficulties.

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	Comments

Diary 2 (4^{th} grade, October)

√ The following frieze is proposed and pupils are asked to <u>describe</u> it on their workbook.

●●○○○○●●○○○○

Pupils' proposals are transcribed on the blackboard. The majority of descriptions underline the alternate presence of the pair and of the set of four objects.
Two examples:

(a) 2 black, 4 white, we need two stamps:

●● ○○○○

(b) We need two stamps:

●●○○ ○○●●

«One stamp is enough!» (authors of previous stamps agree too).
A pupil goes to the blackboard and draws:

(c) ●●○○○○

✓ «Which of these solutions are more helpful if you are to search for the colour of the 36^{th} spot?»

He chooses solution (c) and writes on the blackboard: 2 + 4 + 2 + 4 + 2 + 4 and then, after a moment of silence, specifies «And I stop»

The author of (b) chooses solution (c) too and proposes the calculation '6 times 6' and she says the 36^{th} spot is white.

√ «What about the 38^{th}?»

«6 times 6 plus 2. The second black»

√ «And how about the 92^{nd}?»

The class is enthusiast about the difficulty of calculations, but many pupils do not manage. They are brought to think about the fact they are working with multiples of 6.
The activity is hectic. After a while the solution arrives:

«6 times 15 plus 2. It is the second black» **see Note 1**

→

Note 1: Multiplication and division

A recurrent problem in the initial phases of the activity is brought in by two interventions ("6 times 6 plus 2. The second black" and "6 times 15 plus 2. It is the second black". This problem is slowly overcome through <u>collective</u> reasoning, purposefully piloted by the teacher: pupils are not able to model the situation by means of division.
They spontaneously express their ideas in terms of multiplication since this reproduces their calculations sequentially: they counted stamps until they got close to the place to which the question refers (in this case "what about the 38th?" And "what about the 92nd?") and therefore they have added the spots missing within the last stamp (incomplete).
'Seeing' division is complex because it involves a move to a metacognitive *level. There is a change in point of view: <u>writing</u> must no longer* tell *, although synthesising, about a concrete way of counting, but rather <u>represent</u> the outcome of a conceptual jump. This jump is what enables pupils to understand that, in order to answer the question, they must initially count how many 'stamps' are in the ordered number and then carry out a di-*vision.
*The development of this situation is described in **Note 2: The importance of remainders.***

Activities suitable for classes	*1*	*2*	***3***	***4***	***5***	*1*	*2*	*3*	**Comments**

→
Discussion leads to reflect on the fact that in order to discover the structure of the frame one must not focus on the alternating differently coloured spots, but rather on repetition of the stamp **1**.
Another frame is proposed and the colour of the 59th spot is asked:

●●●○○○○●●●○○○○

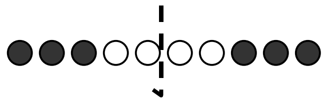
A pupil proposes a misleading solution:
«You make a stamp with three violet and two pink spots and then you turn **2**»
We change the stamp, always putting an odd number of spots; again the colour of the 59th spot is asked.

●●●○○○○●●●○○○○

After some errors in describing the frieze orally, the answer is given verbally:
«7 times 8 plus 3… the third black»

__1.__ About this, see the next Note 3: Different perceptions of a frieze

__2.__ The pupil saw an axial symmetry in the first group of 10 beads:



But she did not check whether the intuition works in the rest of the frieze (which actually does not happen).
Anyway, it is a local perception and therefore not a productive one in terms of identification of the generating law for the sequence.

Note 2: The importance of remainders

The role of the stencil is to favour the identification of the regularity underlying the generation of the sequence, thus allowing the identification of the symbol attached to a certain position. We deem useful to clarify for the reader modalities in which the class can be guided in this search.
Let us consider the following frieze, having a module made of 8 elements: 5 stars and 3 moons.

☆☆☆☆☆☾☾☾☆☆☆☆☆

It is easy to see that the 7th symbol is a moon; the 11th is a star.
In order to find the 26th symbol we must draw on the concept of division, identify the module and divide 26 by the number of its elements, in our case 8.

→

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	Comments

→
From division we get quotient 3 and remainder 2. These data are to be interpreted as follows: 3 represents the number of complete modules that precede the 26^{th} symbol and the remainder 2 points to the position of the symbol inside the module.
Concluding: the symbol is in the second place of the fourth module and therefore it is a star.
A second example, with a much bigger number, as pupils like when they have understood: what is the 1999^{th} symbol?
1999 : 8 = 249 with remainder 7
It is thus the 7^{th} symbol in the 250^{th} module, that is a moon.
In actual fact what counts is not only the *quotient*, but also the *remainder*, and this is a non trivial finding for pupils.
The next step is to identify 'which' star and 'which' moon and it is exactly the remainder that allows pupils to find out that in the first case it is the 2^{nd} star, in the second case it is the 2^{nd} moon.
In the case of remainder 0, once indicated quotient with q, the symbol is in the last place of the q-th term. For instance, in the case of the 24^{th} symbol, we get remainder 0 and quotient 3, ad therefore it is the last of the third module. Again, in the case of the 1992^{nd} symbol, with remainder 0 and quotient 249, it is the last symbol of the 249^{th} module.

Diary 3 (4^{th} grade, October)

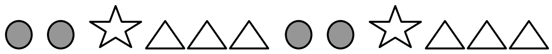
A game is proposed in which <u>Brioshi</u> sends the following frame.

√ «Which drawing do we find at the 82nd place?»
The class identifies the module, which is made of 8 drawings in this case.
«82 is 8 times 10 plus 2 and it is the second star 3»

3. *When the number of the position is small pupils spontaneously carry out mental calculations and think in terms of 'multiplication'. Pupils should be led to understand that it is better to think in terms of division. It might be helpful in this case to work with big numbers, since mental calculation is not possible and the use of division is favoured.*

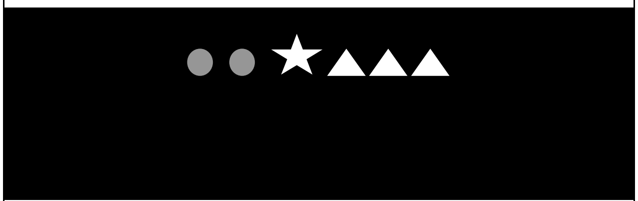
Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	Comments

Diary 4 (5th grade, December)



√ Pupils are asked about the 63rd symbol. Many pupils tend to focus on shape and colour again **4**, some give answers without thinking. The teacher clarifies that in this moment the important thing is not to find the answer but rather to be able to explain how they can organise themselves to answer.

A pupil understands that it is convenient to identify the stencil made of six drawings: two circles, a star, three triangles. She goes to the blackboard to highlight it.



The idea of the stencil works; pupils talk about «repeating the stencil»; the concrete nature of the gesture (repetition), together with the hand that mimes the movement take over.

√ «But when you say you repeat the stencil, do you understand what you are doing from a mathematical point of view?» The class looks puzzled and the teacher asks whether they ever heard about 'multiples'. Of course there are positive answers.

A pupil finally understands «But they are multiples of 6!»

«The stencil is made of 6, then I repeat it and count symbols»

√ «Good! And would you be able to tell me which is the 20th drawing then?»

Some keep counting symbol by symbol.

«I count by 6 until I can and then add …»

→

***4.** See Note 3: Different perceptions of a frieze.*

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	**Comments**

→
A small group elaborates on this reasoning «I do 6, 12, 18 and add 2 which is 20»
√ «Would you be able to transcribe your reasoning for Brioshi?»
Some pupils identify it.
One of them goes to the blackboard to write it:

6 × 3 + 2

«So the second drawing is a circle!»
√ «What about the 57^{th}?»
«6 × 9 + 3 **5**: It is the third drawing, it's the star!»
√ Pupils are asked to focus on the search for the number of times the stencil is used. Reasoning is around the fact that using times tables it is easy to find the drawing if its position is given by a small number. Things are more complex if the bead is in a position with a very big number. How can this be done?
«For example 1548. I take 2 out and get to 1546 which is in the times table of 6»
√ «How do you know that such a big number is in the times table of 6?»
The pupil cannot answer. The class is stuck.
√ A small number is proposed again: the 13^{th} symbol.
«6 × 2 + 1»
√ The teacher tries to suggest division «What are you looking at? How many times… »
«The stamp is made of drawings. We must look at how many times we must repeat it »
√ «And so? What operation do you need?» Perplexity.
√ Pupils are asked to focus on the search for the number of time they use the stencil. After a while they are asked about the 3576^{th} symbol (*drawing* in pupils' slang).
→

5: *Pupils keep using multiplication. About this see Note 1. Multiplication and division.*

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	Comments

→
A group of voices «You make a division!»
«I thought that if it must stay in it, I do 3576 divided by 6»
Classmates understand they are in front of the solution. The division 3576 : 6 = 596 is made at the blackboard and they find out that it is a triangle. Discussion leads to conclude that the key point is not quotient but rather remainder of the division, which represents the position of that particular figure within the stencil.

2. Words and regularities

An activity analogous to previous ones is based on letters rather than symbols. If the stamp is a word the exploration might be easier than the one with friezes because meaning, immediately perceived, helps in identifying regularities. If repetitions of meaningless groups of letters are proposed, then difficulties are analogous to those met while working with friezes.

Diary 5 (5th grade, December)

A pupil has an advertisement stamp in his pen case, made of a rubber stripe, rotating around an inked pin, similar to a tank's caterpillar, which, dragged on a sheet of paper, indefinitely prints the writing CODFACE:

CODFACECODFACECODFACECODFACE

√ The teacher takes the chance to launch the problem «What letter will be at the 244th place?»
«CODFACE is 7 letters, if I make 244 divided by 7 I get 34 with remainder 6»
«Then we must look at the sixth letter, which is the second C.

Activities suitable for classes	1	2	**3**	**4**	**5**	1	2	3	Comments

Diary 6 (5th grade, December)

Pupils are having fun searching for regularities in groups of words. The analysis of their names, repeated infinitely, is proposed. We start from STEFANIA:

STEFANIASTEFANIASTEFANIASTEFANIASTEF

√ «What letter will be at the 150th place?»

Pupils count the name's letters in a hectic way and their answer overlap

«They are eight! 150 divided by 8!»

«And then you look at the remainder!»

«If I do 150 : 8 I get 18 with remainder 6, so when I get to the 150th letter I have written STEFANIA 18 times, and I will be at the 6th letter of the 19th name, and that is N »

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

Second phase: from series to sequences [6]

3. A first necklace: 2 white beads, 5 black beads

The activity starts from the study of a beads necklace, with no defined length, in which 2 white beads and 5 black beads alternate. The necklace can be constructed either with big beads or with coloured pieces of pasta, threaded in a string. In order to suggest the concept of 'necklace with non defined length' the teacher might possibly keep the necklace partially hidden in her fist and let pupils see only twenty initial beads or so. In this way the first pair of white beads will be defined as the 'beginning of the necklace'.

○○●●●●●○○●●●●●○○○○○ …

In the first phase work will be aimed at the identification of the colour of beads located in particular positions, gradually increasing the number identifying the position itself. For example:

- What colour is the 12^{th} bead?
- What colour is the 35^{th} bead?
- What colour is the 123^{rd} bead?

As long as they work with visible beads pupils tend to actually count beads; for instance they easily see that the 12^{th} bead is black. As the number increases though, a strategy becomes necessary, because the 35^{th} bead is hidden inside the fist. Finding a strategy is not easy either for pupils or, often, for the teacher, usually not used to working in situations relating to search for regularities. (see **Note 3**).

6. *As we have anticipated in paragraph 6. 'Didactical aspects', an arithmetic progression is a sequence of numbers characterised by the fact that the difference between a term and its antecedent is constant. In operative terms we might say that a progression is constructed starting from a term and adding a fixed number.*
If the initial term is 3 and the fixed number is 5, we get the progression :
3; 8; 13; 18; 23; 28; …
From now onwards we will keep talking about 'sequences' as general environment, but in fact we will explore arithmetic progressions.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

Note 3: Different perceptions of a frieze

It is worth dealing with the basic difficulty pupils meet when they need to identify a module of a frieze or, more generally, when they search for regularities in the structure of a sequence. This difficulty can be located at the level of the *initial perception* of a frieze or, as in the case we are working on, of a necklace.
If the observer's spontaneous perception, as it is often the case, captures the alternating groups of elements, also different as concerns numbers, colours and shapes, then the search (for instance of the colour of a hidden bead) is lost in this infinite repetition. In this case some regularities are grasped but they sort of 'break down' in an unproductive way among different groups of elements.
For example, in the case of the necklace in the previous page

○○●●●●●○○●●●●●○○○○○…

Observation allows pupils to grasp repetitions of white beads and that of black beads as distinct facts, and when the colour of an invisible bead is asked, pupils start to focus on white beads, and then on black ones and again on white ones, in an attempt which becomes a frustrating operative block.
This fragmentation hinders the development of exploration and is an obstacle to the identification of a *structure*, i.e. the generating law for the necklace, and consequently of the net of relationships linking components of the considered sequence .
Identification of the regularity comes from a *rupture* with a perception of this kind and represents the fruit of a *metacognitive* operation: information provided by perception must be elaborated *changing one's point of view completely*. In other words, one must learn to identify the *module* of the sequence.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

Diary 7 (4^{th} grade, November)

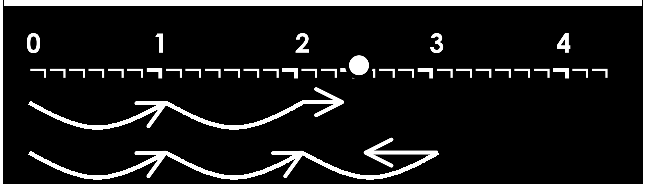
We want pupils to describe the necklace:

○○●●●●●○○●●●●●○○○○○

Descriptions are written on the blackboard.

(a) The stencil starts with two white and ends with 5 black
(b) 2 whites and 5 blacks to infinity
(c) You start with 2 white beads and then 5 black and you carry on like this
(d) The stencil is 2 white beads and 5 black ones to infinity
(e) Stamps made of 7 beads, 2 white and 5 black

Discussion enables a clarification of differences between descriptions; (a), (d) and (e) highlight 'stamps' and their repetition, whereas (b) and (c) underline the infinite alternation of the two groups of beads. Many pupils do not grasp the difference between the two points of view, maybe because the authors of (b) and (c) claim they have said the same thing but were not able to express their thoughts clearly. **7**
√ Definitions are read out loud; then the colour of the 50^{th} bead is asked. Two strategies are proposed (one of them unexpected) translated into formulae on the blackboard:
(f) 7 × 7 + 1 (g) 7 × 8 - 6
√ «How would you explain to a pupil from another class these two formulae ?»
In order to stimulate an answer the bus metaphor is brought in. The following drawing representing the route is made on the blackboard. '0' represents the terminus and successive stops are numbered as the drawing is being 'told'.



→

7. *Pupils tend to use terms such as 'stencil' and 'stamp' because they refer to concrete objects and to experiences they carried out in the preliminary phase. These are important terms because they convey a key concept for the identification of regularities. It is nevertheless important that language is gradually refined and pupils move to a more correct use of the term 'module'.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
√ «Imagine that stops are located at each multiple of 7. A person, to go to a certain place between two stops can decide whether to get down the bus at the preceding or at the subsequent one, depending on which one is closer to the place he or she needs to go»
The class grasps clearly the difference between (f) and (g).
«You must search for the next multiple and then add or take out the number that works»

Diary 8 (4th grade, November)

○○●●●●●○○●●●●●○○○○○...

√ «What colour is the 35^{th} bead?»
«7 times 5 is 35… the 35^{th} bead is black»
√ «what about the 37^{th}?»
He goes to the blackboard and writes while talking aloud …

$$7 \times 5 + 2$$

… and concludes «the second white one»
√ «What about the 92^{nd}?»
Pupils are suggested to reflect on the fact that they are working with multiples of 7. After a while the written solution arrives together with the answer.
«7 times 13 plus 1. It is the first white one»

Expansion 1: Towards generalisation of representations of division

About the writing 7 × 5 + 2. It is important that pupils view division as a binary operation, acting on a couple of numbers (dividend and divisor) giving another couple as result: quotient and remainder. It is equally important to induce students to express in many ways the links between dividend, divisor, quotient and remainder. For instance, division between 15 and 6 leads to quotient 2 and remainder 3. Relationships linking the four numbers can be variously represented:

- $15 = 2 \times 6 + 3$
- $15 - 2 \times 6 = 3$
- $(15 - 3) : 6 = 2$

With older pupils one can get to generalisation: given two natural numbers a and b, with b >0, and let q and r be respectively quotient and remainder, we can express their mutual relationships in several ways:

- $a = q \times b + r$
- $a - q \times b = r$
- $(a - r) : b = q$.

This exercise in transcribing, if done since the early approach to division, will avoid well known rigidities in coding formally this link and will possibly favour flexibility in realising algebraic transformations.

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

Diary 9 (4th grade, November)

> The previous diary is continued.
>
> ○○●●●●●○○●●●●●○○○○○
>
> ➥ «The last bead, the seventh, is black and if I carry on like this I find out that all multiples of 7 correspond to black beads »
> √ «What colour will be the 123rd bead? Why?»
> ➥ «I saw how much you get dividing 123 by 7. It is 18 with remainder 3. 18 is not that important, but the remainder is, because it means that I must see what colour is the third bead and that is black »
> √ «What if the remainder is zero?»
> ➥ «If I divide by 7 and the remainder is zero it means that the number is a multiple of 7 and since every 7 beads the bead is black then remainder zero means black bead » **8**

__8.__ The approach to the problem changes from the 4th to the 5th grade: in the former case pupils use the direct operation (multiplication), in the latter the inverse operation (division). This move might be interpreted as use of a more developed form of thinking.

The activity enacts reasoning about regularities and also about the concept and meaning of division, about properties of multiples and divisors, about remainder classes and prospectively about modular arithmetic.
It is important to point out, as indicated in **Note 2**, that the result of the problem is not only the quotient of the division between the number identifying the bead and the cardinality of the module, but it is also its remainder.
This activity is important because it gives a chance to reflect on the often neglected meaning of *remainder* of a division and contributes to drawing a distinction between calculation and interpretation of the obtained results.

4. In what position is the black bead?

A whole new problem situation is posed:
«In what position is the 15th *black* bead?»
This is different to asking:
«In what position is the 15th bead?»
We deem important to stop and reflect on this question before going into the activity.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

Note 4: Reflections about the necklace

The initial question is:

«In what position is the 15^{th} *black* bead?»

In this we refer, besides colour, to *two* positions, a known one (the 15^{th} bead) and an unknown one («In what position…»).
It is better to represent them:

○○●●●●●○○●●●●●○○●●●●●

↑ ↑

15^{a} bead 15^{a} bead

In order to find the position of the bead in the necklace one refers to the module, but in this case you find the n-th bead and not the n-th bead of that particular *colour*.
Now – and it is here that a new restructuring of the field is necessary – modules cannot be '<u>opaque</u>' any longer, rather they must be viewed again as *sets of beads of two different colours*. The steps can be schematised as follows:

perception of the necklace
as repetition of an 'opaque' module

↓

perception of the necklace as *repetition of a* '<u>*transparen*</u>*t' module*
made of 2 *sub-modules*
respectively of 2 white beads and 5 black beads

↓

perception of the *repetition*
of the sub-module made of *5 black* beads

The 15^{th} black bead is then in the *third sub-module* of *black* beads and the search for it is represented by the <u>result</u> of this <u>process</u>:

15 : 5 = 3 with remainder 0

the requested bead is the 5^{th} black bead of the third module. **9**
We propose another question without a visible counterpart in the necklace's drawing:

→

9. *A procedural form of reasoning might be: each module has two white beads, hence the total number of white beads before the* 15^{th} black bead *is 6. Therefore this is at the* 21^{st} *place (6 + 15 = 21).*
A global type of reasoning might be: the 15^{th} *black bead is the* last in the third module. *Since the module has 7 beads, then the bead is at the* 21^{st} *place because 21 = 7 × 3.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
«In what position in the necklace is the 89th *black* bead?»
The answer is:
89 : 5 = 17 with remainder 4
The requested bead is then the 4th of the *18th sub-module*.
But this answer is not enough.
In order to find the position of a bead of a certain colour in the necklace you must understand that in fact the sub-module corresponds to the module and therefore you must not think about the 5 black beads of the *sub-module*, but rather consider the 7 beads of the *module* again.
The full answer enabling identification of the position of the 89th black bead in the necklace can be found in the following way:

89 : 5 = 17 with remainder 4
$7 \times 17 + 2 + 4 = 125$

In this last formula we added also the *two white beads* that precede the *four black* ones of course. Sub-module and module must be controlled at the same time; the 4th black bead in the sub-module is the 6th bead in the module. The 89th black bead is in the 125th position.

Expansion 2: <u>Coding</u> the relationship between place and symbol

A possibility is to get to coding the general relationship between *number of place* and *correspondent symbol* in friezes or necklaces having analogous structures, but different length of the complete module.
For this pupils are led to observe at least three different sequences to point out that, beyond differences (type of symbol, length of module and types of symbols in modules), they can be framed in a single scheme.
For example, let us consider the following sequences:

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

(a) ●●●◇◇●●●◇◇●●●◇◇●●●◇◇●● …

(b) ☆☆☆○☆☆☆○☆☆☆○☆☆☆○☆☆☆○☆ …

(c) ▲▲▲□□○□□▲▲▲□□○□□▲▲▲□ …

The length of the module is different in the three cases

●●●◇◇ ☆☆☆○ ▲▲▲□□○□□

But the three situations can be unified representing the length of the module with one same letter, independently on the number or type of beads that constitute it.
Comparing different situations of this type pupils get to unify them, by giving the same *name* to elements that play the same role.
Representing with:

- *a* the number identifying a symbol in an indefinite sequence;
- *b* the length of the module, i.e. the number of symbols that compose it;
- *q* the quotient of the division between *a* and *b*;
- *r* the remainder of the division. **10**

So *q* indicates the *number of complete modules that precede the bead in place a*

and

r indicates *the number identifying the bead in place a which is the last of the q-th module if r=0, otherwise it is the symbol in place r in the module that follows the q-th one in the indefinite sequence.*

This relationship can be expressed through the equalities:
$a = q \times b + r$ $\quad a - q \times b = r$ $\quad (a - r) : b = q$ **11**
each of which have a different subject and are differently readable in terms of the problem.

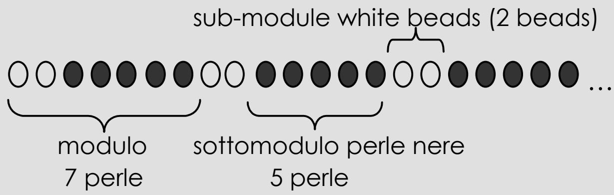
10. *These aspects will be analysed in* ***Expansion*** *2.*

11. *Learning to recognise equivalent but formally different writings, means to recognise their logical structure, that is the relationships linking its elements.*

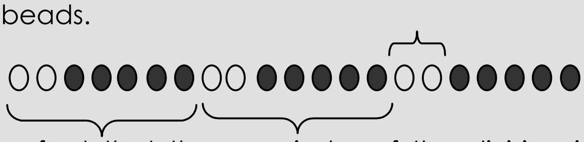
Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

Expansion 3: The position of any odd bead

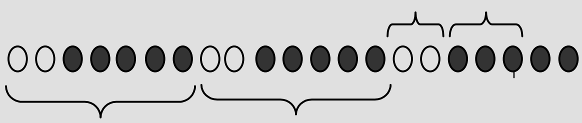
With junior high school pupils (mainly 8th grade) you can go farther down the generalisation route.
Consider the following necklace again:



Suppose you want to *find the position of any bead, either black or white*.
At first it is convenient to reflect on a particular situation: *find the position of the 13th black bead*.
Let us analyse the problem.
In order to calculate the number identifying the 13th black bead in the sequence, we first need to identify in which module it is. To determine this it is enough to divide number 13 by 5 (length of the black beads sub-module).
This division gives 2 as quotient and 3 as remainder.
To be able to interpret these data in relation with the sequence, one must see the two black beads sub-modules as immersed in the modules constituting the necklace having length 7.
Therefore in the necklace, before the 13th black bead, there are two complete modules, i.e. 2 × 7 beads.



The fact that the remainder of the division is 3 tells us that our bead is the third among the 5 black ones, but these are preceded by 2 white beads. Hence the 13th black bead is the bead in the place (2 x 7 + 2 + 3), that is the 19th.



→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
Were the black bead the 15th, dividing by 5 (length of the black beads sub-module) you get quotient 3 and remainder 0. In this case the number identifying the bead in the sequence is given by the number of beads in the 3 modules (7x3=21), the 15th black bead is the 21st in the sequence.
Position in the necklace is the 19th.
This procedure can be schematised by introducing letters to represent the various elements involved. Let us name:
- n the number identifying the black bead among black ones;
- s the length of the black beads' sub-module;
- b the length of the non-black beads' sub-module;
- m the length of the whole module;
- q the quotient of the division between n and s **12**
- r the remainder of the division between n and s **13**

The procedure can be summarised as follows:

number n of black beads up to the n-th

|

divided by the number s of the black sub-module's beads

↓

you get:

number q = (n – r) : s of modules preceding that with the 13th black bead

|

multiplying q by number m of beads in the module

↓

you get:

the number q × m of beads in modules preceding the n-th black bead

case r = 0
Position of the n-th black bead in the sequence is exactly q × m

→

12. *by n : s we mean the quotient.*

13. *Quotient q can be also represented as (n - r) : s.*

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

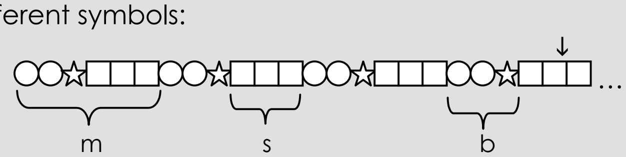
→
case $r \neq 0$

if we want to find position
of the n-th bead in the sequence
can be obtained by adding to $q \times m$
the number of beads of the white sub-modules
plus the remainder r.

The literal translation of the sequence described above is written for Brioshi:

$$p = (n - r) : s \times m + b + r$$

The formula can be described as follows: 'the bead's position can be found dividing the number of black beads by the length of their sub-module, taking the quotient and multiplying it by the module's length, after that …' pupils soon realise that algebraic representation is far more convenient than representation in natural language.
In the case of a module with more than two types of beads the same reasoning still holds: it is enough to interpret the module in terms of 'black bead' and 'non black beads'.
For example, in a sequence like the following, in which the module is made of repetitions of 3 different symbols:



if we want to find the position P of the 11th square (marked by the arrow) we can apply the same formula (where 'b' represents 'non squares'):

$$P = n : s \times m + r + b$$

$$P = 11 : 3 \times 6 + 2 + 3$$

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

Diary 10 (4th grade) **see Note 5**

○○●●●●●○○●●●●●○○○○○

√ «Attention please… I am going to propose a more *difficult* problem now … [14]. In what position is the 22nd **black** bead?»

Prompt and hectic answers are given.

- «5 times 3 minus 1»
- «7 times 2 plus 1!»
- «No! 7 times 3 plus 1! It's white!»

Control of the last formula enables pupils to understand that they were not seeking the 22nd *black* bead but rather simply the 22nd bead.

- «7 times 4 minus 6!»
- «That is still the 22nd bead »
- Pupils made very long drawings. Beads drawn on the blackboard are counted and they find out that the 22nd black bead is at the 32nd place.

Finally we get to correct proposals, which are written on the blackboard.

7 × 4 + 2 + 2 **7 × 5 – 3**

Diary 11 (5th grade)

○○●●●●●○○●●●●●○○○○○

√: «In what place is the 10th white bead?» [15]

- «It is the 30th place: I counted on my drawing [16]»
- « I did the 2 times table and I saw that it is in the 5th group »
- «I look at the blackboard *[22 beads are drawn and the last one is the 7th white]*. If the 7th white bead is at the 22nd place, the 8th white will be at the 23rd. There are still 2 to the 10th, I add 7 and I get to the 30th position »

→

Note 5: Invitation to the reader

Diaries between 10 and 12 refer to the necklace we just analysed and in some cases it is not that easy to follow pupils' interventions – be they correct or wrong- and their conclusions.
The reader is invited to answer himself the initial question posed in each case and evaluate strategies and possible difficulties, maybe linked to what has been said about identification of modules and sub-modules.

14. *The sentence should not be interpreted in its literal meaning. There is a complicity atmosphere in the class and the real meaning is: "I challenge you but my voice intonation is playful and it is not that difficult as I seem to be telling you". Pupils like the situation because of its risky features, but they know that they can manage and the teacher will work with them.*

15. *In exploring the necklace there might be interferences between perception and reasoning, since the module contains unbalanced numbers of differently coloured beads, mainly against the perceptively weakest colour. This might cause difficulties for pupils. Should this happen, the teacher might want to work with a more balanced module (such as 3 and 4).*

16. *It is a usual strategy. If the number is not too high there are some pupils who patiently draw 100 or 200 beads and then count them.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→
«There are 4 groups of 7 before the 10th bead: 7 plus 7 *(thinking)* plus 7 plus 7. Yes, it's true, it is at the 30th place »
√: «Now let us look for the position of the 100th white bead »
«The 100th white bead is the second in the 50th couple. This means that there are 49 couples before that. Then I must add 2 » **17**
«If I seek black beads I do 49 times 5 which is 245. So before the 100th white bead there are 245 black ones. I add 100 white ones to them and I get 345 beads. The 100th white bead is at the 345th place **18**»

Diary 12 (5th grade)

○○●●●●●○○●●●●●○○○○○

√ «In what position is the 3rd white bead?»
The answer is given promptly «It is the 8th bead, I counted and I saw that the 3rd white bead is at the 8th place!»
√ «In what position is the 10th white bead?»
Pupils think for a longer time because they must 'add' non visible beads.
«I counted the 10 white beads and then I added the black ones in the middle: 4 groups of 5, so the 10th white bead is at the 30th place »
«Yes I did that too: 2 times 5, I multiplied by 4 and found 28. Then I added other 2 white beads, the 9th and the 10th and so I get to the 30th place»
√ «What about the 31st white bead? Where will that be?»
«If the 10th white bead is at the 30th place, the 30th white bead will be at the 90th place and so the 31st white one is at the 91st place »
It seems to work but some pupils disagree.
«No, because the 31st white bead does not come straight after the 30th, there are the 5 black ones in the middle to count **19**»
→

17. *The pupil focused only on pairs of white beads, as if black beads were not there. The hazy conclusion of his reasoning confirms the scarce control he has of the situation.*

18. *Pupils elaborate their reasoning on the basis of two submodules which, summed up, give the requested position.*

19. *Some pupils enact a sophisticated strategy at this moment, of a proportional type. Their intervention refutes the previous one and proves that it was not correct. This is a meaningful example of how class discussions contribute to conceptualisation and favour a social construction of knowledge.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

«We cannot go by 10s, we do not have modules of 10!»

For a while pupils work individually. After that they start talking and <u>exchanging ideas</u> .

«The bead is at the 106th place: I drew and counted them one by one »

Classmates peep in the pupil's workbook.

A pupil makes a remark which allows the class to go ahead.

«Why can't we count them by 7, since they are modules of 7?»

«With 31 white beads I construct 15 pairs of white beads and then the 31st is the one coming straight after »

«It's true, I did 15 × 7 = 105, and then the 31st white bead will be the 106th in the necklace »

√ «In what position will the 43rd black bead be?» **20**

«Black beads are grouped by 5. With 43 black beads I make 8 groups of 5 beads and I have 3 left, therefore it is the 3rd black bead after 8 complete groups » **21**

Proposals made by a group of students enable an organisation of the latter proposal in an understandable form on the blackboard:

5 × 8 = 40 black beads in 8 groups
2 × 8 = 16 white beads in 8 groups

And they conclude «Altogether, 8 groups give 56 beads, plus the two white ones, and they become 58. The third bead coming after these is the 61st » **22**

20. *There is a notable conceptual jump. A field restructuring is necessary together with a move from the white sub-module to the black sub-module.*

21. *The pupil focuses on co-ordination between black beads' sub-module and module but a part of <u>argumentation</u> is missing: she does not realise that she should add also white beads which are in the module.*

22. T*he two expressions written on the blackboard are characterised by perception of the two sub-modules which are alternatively repeated. In the conclusion there is a reference to a move from repetition of sub-modules to repetition of the module.*
The following expression is implicit

$$7 \times 8 + 2 + 3 = 61.$$

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

5. Necklaces and the distributive law

Reflection on representations from Diary 12

$5 \times 8 = 40 \qquad 2 \times 8 = 16 \qquad 7 \times 8 = 56$

gives cues to tackle or deepen the study of distributive law, within a context which makes its meaning clearly visible. Pupils might be invited to reflect on the equality coming from the same 'effect' of the two routes followed to identify the bead:

$$5 \times 8 + 2 \times 8 = 7 \times 8$$

and think in terms of another change of representation:

$$5 \times 8 + 2 \times 8 = 7 \times 8 = (5 + 2) \times 8$$

6. A second necklace: 2 white beads, 2 black beads

Necklaces with an equal number of beads of the two colours can be proposed; for instance, necklaces in which pairs of white beads alternate with pairs of black beads. The increased regularity makes the necklace more difficult to be analysed, because focus is posed on the alternation of the two sub-modules. The field (in a *Gestaltic* sense) is perceptively *neutral* and slows down the restructuring process. For these reasons this is a favourable chance for verifying and reflecting on the class 'achievements up to that point.

○ ○ ● ● ○ ○ ● ● ○ ○ ● ● ○ ○ ● ●

The first reflection is about a bead's position within the pair.

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

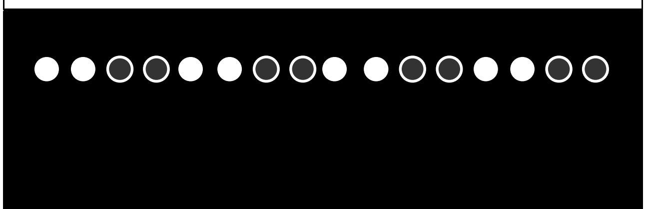
Diary 13 (5th grade, November)

√ «Is the 24th bead the first or the second in the pair?»
The class thinks.
«It is the second»
√ «Is the 9th the first or the second bead in the pair?»
«It is the first»
√ «How can you guess?»
«If the number is odd the bead is the first in the pair, if it is even, it is the second »

Hence reflection moves to the alternation of colours:

Diary 14 (5th grade, November)

«Weird, since the 10th bead is white I thought the 20th would be white, but it is black. In the first ten you start from the white bead, in the second ten it is as if you started from the black bead »
The necklace is drawn at the blackboard:



«The 20th bead is black, I can see it. So in every ten colours alternate »
Discussion leads to increasingly complex answers:
«Yes, the even ten are black, the odd ten are white » [23]
«You must look at the ten's figure»
«If the number of the bead ends by zero, you must study the ten's figure to identify the colour: odd white, even black »
«The beads corresponding to a number ending by two or more zeros are always black »

23. *Pupils restructure their view of the necklace in groups of 10 beads and talk about even and odd tens, thus grasping the alternation of the first bead's colour, with which groups are linked in the necklace.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

Work continues with the search for beads' colour in relation to the position within the necklace:

- What colour is the 10th bead?
- What colour is the 35th bead?
- What colour is the 123rd bead?

Diary 15 (5th grade, December)

○ ○ ● ● ○ ○ ● ● ○ ○ ● ● ○ ○ ● ●

«The 10th bead is white, because I tried to count, the 10th bead came out white »
«I started with two white beads. I counted the groups of two, the pairs we need to get to 10. I found 5 because 10 divided by 2 is 5. then I counted: white-black-white-black-white: the 5th pair is white, the 10th bead is white »
«I think the 10th bead is white because 10 is even »
«No, it's not because of this, because there are even beads which are black, just look »

Pupils need to get to identify the 4 beads module, a necessary step toward generalisation (the necklace is always the same).

Diary 16 (4th grade, December)

○ ○ ● ● ○ ○ ● ● ○ ○ ● ● ○ ○ ● ●

«We might do the 4 times table; with 4 you always get the black one »
Pupils check that the hypothesis works.
√ «Can you find the 25th bead's colour?»
A pupil goes to the blackboard and goes on by successive steps. He writes:

16 + 4 = 20

→

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→
He realises that it is not enough and writes under that:

$$20 + 4 = 24$$

It is not enough yet and he adds:

$$24 + 1 = 25$$

He finally finds that the bead is white.
√ «How could we describe the rule?»
«In this necklace any multiple of 4 is the second bead of the black pair »
√ «Is there a way to describe the 25th bead's colour in <u>mathematical</u> language for Brioshi, using the rule Rossella stated?»
The following writing is collectively reached and reported on the blackboard:

$$4 \times 6 + 1$$ [24]

√ «And how do you find the 38th bead?»
The class shows to be sure about this and they write:

$$4 \times 9 + 2$$

During discussions strategies deriving from diverging points of view may emerge: these can be very interesting, although sometimes they are not immediately understood.
An example is reported in next Diary.

Diary 17 (5th grade, January)

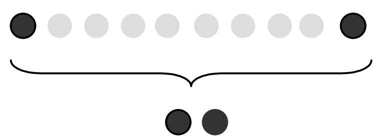
√ «What colour is the 32nd bead? Why?»
Using the drawing to explain the followed strategy:

○○●●○○●●○ ○●●○○●●

«The first two white beads represent the endpoints of the first ten, starting with white and finishing with white, the second pair of beads represents the second ten, starting with black and finishing with black, the third pair is the third ten and I go on for two beads more and I find the black bead. So the 32nd bead is black » [25]

24. *As pointed out in Comments 3 and 5 , also in this Diary we find* Multiplicative *representations; the class must still appropriate the concept that the remainder can be found only after carrying out a division. Probably small numbers favour the use of multiplication*.

25. *The pupil, while reading her drawing, restructures the field in a distinctive way: she makes beads between the endpoints of a ten 'opaque' and constructs a compact model of tens, made of a pair of beads of the same colour as the endpoints of the ten it represents.*



She counts pairs until she approximates the sought number to a maximum of 10 (in this case 3); then she gives the bead its initial role as unit.
We think that this is a good example of how a rich activity can stimulate creativity, rationality and language at the same time.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

7. The second necklace: where is the bead?

Let us move to the last part of the activity with the new necklace. The activity is the same as that proposed in the Third phase. We will try and link either white or black beads to the position they have inside the necklace. We will pose questions such as:

- In what position is the 3rd black bead?
- What about the 3rd white bead?

Pupils are now ready to move to generalisation.

Diary 18 (5th grade, February)

○ ○ ● ● ○ ○ ● ● ○ ○ ● ● ○ ○ ● ●

√ «In what position is the 3rd black bead?»
Answer to this question is simple because you can see straight away that the bead is in the 7th place.
√ «In what position is the 9th black bead?»
Yes, it is in the 19th place.
A pupil proposes the calculation 9 + 9 + 1 to find it
√ «In what position is the 15th black bead?»
This is more difficult because the 15th black bead is not drawn on the blackboard.
√ A proposal is made to enact an organised search through the construction of a table (pupils know this instrument). The first column will report the ordered number of the black bead and the second column will report the related position in the necklace.

n. of black bead	position in the necklace
1	3
3	7
5	11
7	15
9	19

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→
√ Pupils are invited to find out how you pass from numbers in the left column to numbers in the right column. Shortly they find out the 'rule' and write it on the blackboard.

Table 1

n. of black bead	position in the necklace	'rule'
1	3	1 × 2 + 1
3	…7	3 × 2 + 1
5	11	5 × 2 + 1
7	15	7 × 2 + 1
9	19	9 × 2 + 1

A choral conclusion is reached: the 15th black bead is in position

$$15 \times 2 + 1 = 31$$

√ «Can you find out in which position is the 30th black bead?»
«It is in the 61st position!»
A pupil has made a very long drawing and shows it to the others: the 30th black bead is not in the 61st but rather in the 60th position!
The class is disappointed.
The formula they found only holds for black beads which are in an odd number position.
We then construct another table:

Table 2

n. of black bead	position in the necklace	'rule'
2	4	2 × 2
4	8	4 × 2
6	12	6 × 2
8	16	8 × 2

√ Pupils are now invited to describe orally some *General Rules* related to black beads.
«If the black bead's number is odd, its position is twice that plus 1» *[Table 1]*

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→
«If the black bead's number is even its position is the double » *[Table 2]*
√ Pupils are asked to verify through a table whether the rule works for white beads too.

Table 3

n. of white bead	**position in the necklace**	**'rule'**
1	**1**	**1 × 2 - 1**
3	**5**	**3 × 2 - 1**
5	**9**	**5 × 2 - 1**
7	**13**	**7 × 2 - 1**
9	**17**	**9 × 2 - 1**

Once they have verified that the rule changes that table of white beads in even positions is completed:

Table 4

n. of white bead	**position in the necklace**	**'rule'**
2	**2**	**2 × 2 - 2**
4	**6**	**4 × 2 - 2**
6	**10**	**6 × 2 - 2**
8	**14**	**8 × 2 - 2**
10	**18**	**10 × 2 - 2**

Once the table is made the following conclusions are drawn:
«If the white bead's number is odd its position is twice that minus 1»
«If the white bead's number is even, its position is twice that minus 2 »
√ Pupils are asked to translate the Rules in mathematical language for Brioshi, so that the position of any bead in the necklace can be found knowing the colour.
Pupils choose letter 'p' to indicate the bead's position.
We write on the blackboard:

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

black bead	**tab.1**	**odd n. :**	**$p \times 2 + 1$**
	tab.2	**even n. :**	**$p \times 2$**
white bead	**tab.3**	**odd n. :**	**$p \times 2 - 1$**
	tab.4	**even n. :**	**$p \times 2 - 2$**

Third phase

In previous Phases, pupils could play with stencils, manipulate objects, draw them and directly verify the validity of their own hypotheses through a variety of concrete situations (friezes, borders, necklaces). They learned how to explore a *sequence*, from the initial search for elements that define its structure – hence the recognition of its *module*- to the identification of general formulae which relate any element to its *position*.
Now the Third Phase starts, in which the class faces a completely different situation, much more abstract and complex to be analysed: the *arithmetic sequence*.
Let us define briefly the essential features of this change.
Each of the situations pupils faced so far are characterised by an *actual repetition* of the same group of elements *to infinity*. For instance, the module made of three white beads and four black beads can be found *unchanged* in any part of the frieze. There are no 'ongoing' changes: the frieze is an indefinite and 'transparent' sequence of *clones* of the *module*.
In the case of arithmetic sequences this principle is twisted.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

The module is no longer *visible* as before, not even when it has been identified. It is hidden, but the *effects of its application* are visible, and they are the key to recognise the sequence's structure.
Exploration must start from here.
In the necklace, analysis started from the search for the repeated set of elements and help given by *perceptive* aspects was important.
In the arithmetic sequence '2 - 7 - 12 - 17- 22 - …' which will be the basis of next Situations, perception of figurative elements is no longer helpful, since each element represents a sort of *evolution* of the previous element, obtained by applying the module.
So pupils must start from effects and learn how to recognise them.

8. Continuing the arithmetic sequence

The activity starts presenting a particular sequence –arithmetic progression- that pupils must analyse, continue and whose generation rule they must describe. The sequence is:

2 7 12 17 22 …

The answer generally arrives quickly:

… 27, 32, 37, 42, 47, 52, etc.

We might also get different answers, highlighting mistakes or peculiar interpretations.

Diary 19 (4th grade, October)

A pupil proposes the following sequence:

2 7 12 17 72 21 22 11 77 221

√ «Why do you suggest the sequence might be continued in this way?»

«I took 7 and 2 and I did 72»

From further attempts to explain we understand that the girl through she had to combine figures 1, 2 and 7 she identified in the first four numbers of the sequence in different ways.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

Diary 20 (5th grade, October)

A number of different sequences are written on the blackboard:

(a) 2, 7, 12, 17, 22, 24, 31, 43, 60
(b) 2, 7, 12, 17, 22, 60, 61, 62, 63
(c) 2, 7, 12, 17, 22, 37, 42, 57, 62, 77
(d) 2, 7, 12, 17, 22, 27, 32, 34, 41

√ We ask the authors to explain their sequences; explanations are complicated and need teachers' and classmates interventions.
(a) A pupil got numbers after 22 by adding each time 2, 7, 12, 17, 22 in this way: 22 + 2 = 24; 24 + 7 = 31; 31 + 12 = 43; 43 + 17 = 60; etc.
(b) A pupil added the given numbers (2 + 7 + 12 + 17 + 22 = 60) and then added each time a unit;
(c) A pupil observed that the units' figure is alternatively 2 or 7 and that from 17 onwards the ten's figure increases by 1 .
(d) A pupil 'perceived' two different sequences: 2 – 12 – 22 and 7 – 17 and their rule: + 10 and in this way he wrote 27 and 32; then he lost control of the situation and started again to add the first numbers of the sequence: 32 + 2 = 34 and 34 + 7 = 41. **25**

9. Describing the arithmetic sequence

The next step is asking for a description of the sequence.
The following diaries, referring to the same sequence

2 7 12 17 22 27 32 …

provide some good examples of the importance of <u>verbalisation</u> in both refining language and collectively constructing knowledge.

25. *Looking at pupils' attempts (the same thing could be said for adults) we realise that there is not a single way to 'see' a sequence. Infinite ways are possible – not definable a priori as 'right' or 'wrong'- and they depend on observers' sensitivity, fantasy, creativity, curiosity (sometimes also on the fact that the task is not understood). Each of them reveals efforts that we can sometimes understand only trying to interpret hidden intentions of their authors.*
The point is that among all these ways only one is productive in terms of identifying the sequence's structure and it seems that this 'one' is spontaneous only for few pupils. Others must be led to understand how to select that one among the many possible ones and avoid to be attracted by local regularities that do not hold for the whole sequence.
An example: some pupils from the first group perceive the rule '+5'; this leads them to add n + 5 each time, thus keeping control of the meaning of the process constantly.
Differently, pupils from the second group perceive, for instance, regularity of the alternation of final figures 2 and 7 and this leads them to write sequentially pairs of numbers in which the ten's figure increases by one from pair to pair and the units' figure is always either 2 or 7.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

At the end of Diary 22 the class gets to a shared definition, coming from social mediation.

Diary 21 (5th grade, October)

«In the place of the unit's figure we must always have either 2 or 7 »
«Making +5 +5 every two steps we make +10»
«Each time I add 5»

Diary 22 (4th grade, October)

«The difference is +5; units are always 2 and 7; tens decrease»
«From 2 to 7 it is +5»
«Between 2 and 7 the difference is +5»
«Starting from 2 to get to the sequence is +5»
«Between 2 and 7 you add 5»
«The distance between two successive numbers is 5»
«Between each number of the sequence the distance is +5»
«Between each number and the other of the sequence the rule is +5»
«Between each number and its successive in the sequence the rule is +5»
«Between each number of the sequence and its successive the rule is +5»
√ How do we know whether a number belongs to the sequence? Does number 49 belong to this sequence? What about 3512? Do we have information about this number?»
Pupils seem to be unsure.
«3512 belongs to the series because it ends by 2 » **26**
«To belong to the series the number must end by either 2 or 7»
«The units figure must end by 2 or 7 »
√ «Must the unit's figure 'end by'?»

→

Comments:

***26.** The pupils has a correct intuition but cannot express it. This passage is not simple and must be supported through reflection on the reasons underlying that statement. An initial remark might be that the progression is generated starting from 2 and adding 5 to the terms that are gradually obtained.*
This induces to see the subsequence obtained by adding 1 every two steps starting from 2.
By writing number 3512 in the form

$$2 + 351 \times 10$$

it is clear that number 3512 belongs to the sequence and can be reached after 351 double steps, or rather after 35 1 × 2 steps.

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

→
«The units' figure *must be* either 2 or 7»
«To know whether a number is in the sequence the last figure must be either 2 or 7 »
«To know whether a number belongs to the sequence the last figure must be either 2 or 7 »
√ The teacher stimulates the class to substitute the words 'last figure' for a richer terminology on the mathematical plane.
«To know whether a number belongs to the sequence the units' figure must be either 2 or 7 »
AGREED RULE: A number belongs to this sequence if the units figure is either 2 or 7. **27**

Diary 23 (5th grade, October)

√ The teacher asks pupils to communicate to a classmate how a sequence is constructed without writing it. We get the following answers:
«From 0 I always add 5 »
Puzzled pupils «It is not true that we start from 0»
«I add a ten every two numbers»
The class seems to be uncertain because the remark is true but giving the impression that intermediate numbers are 'skipped'.
«I start from any number and I always add 5 »
«It is not true that I start from any number»
«I always add 5 speaking to any number »
«The unit does not change»
«I start from 2 and I always add 5 »
…

***27.** The Diary shows an interesting example of a collective construction of reasoning, leading to the elaboration of a final definition socially shared by the whole class.*
We point out again that activities like this one favour structuring of logical thinking. In actual fact they favour argumentation by pupils who often sense reasons underlying the problem situation they are exploring, but are not able to express them clearly.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

We often get the chance to deepen our reflection on mathematical language, in particular on (more or less elegant) equivalent paraphrases to express one same concept: **28**

(i) For instance the use of 'I add', 'I sum up' or ' I subtract' is favoured instead of generic terms like 'I do';
(ii) Translation of mathematical sentences such as '2 + 4' is analysed and the reached conclusion is that 'I add 2 to 4' and 'I add 4 to 2' are equivalent by the commutative law, but they are not equivalent if we look at the literal translation of the process;
(iii) Pupils' sentences are carefully compared and analogies and differences are highlighted, for instance: 357 and 49562 are numbers
(a) ' that end by 7 and 2';
(b) '… that finish …',
(c) '… that conclude …',
(d) '… that end with the figures 2 and 7',
(e) '… that end with figures 2 or 7',
(f) '… that in the units have either 2 or 7'
and so on.

The difference between 'number' and 'figure' is then analysed because pupils use them unconsciously to propose their sentences.

28. *A deeper analysis of these themes is in Unit 1 of this collection 'Brioshi and the approach to algebraic code'.*

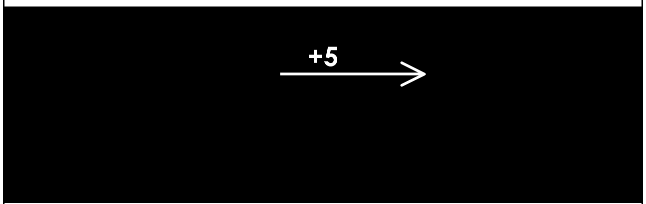
Diary 24 (4th grade, October)

Before specifying the sequence's 'rule' from a mathematical point of view (+5, starting point 2) the class is led to focus on linguistic aspects linked to the definition of the terms 'sequence' and 'rule' (that will be later substituted by 'sequence' and 'step').
√ Pupils are asked to find out the way in which the sequence is generated and to express it.
Pupils talk about: 'series of numbers', 'path of numbers', 'line of numbers' and about 'rule'.
√ They are asked to specify what they mean by 'Rule'.
→

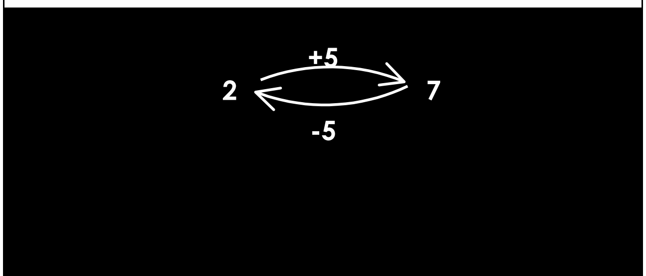
Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

→

- «It is the difference between a number and the other»
- « It is the space between one number and the other»
- The pupil goes to the blackboard and draws an arrow with +5 on the top.



- (going back to the 'sense' of the last representation) «The difference is the path I use to find the result of an operation »
- «The difference is the result of subtraction»
- He writes 2 and 7 on the blackboard and joins them with two opposite arrows; on the top one he writes +5 and on the bottom one –5:



We try to get a linguistically complete definition of the law.

√ «Try to clarify the '+5 law'»

- «Between a preceding and a successive number the rule is +5»
- «Between a number and that coming after the rule is +5»
- «Between a number and its successive the rule is +5»

The last definition is accepted as the most correct one and it is written on the workbook.

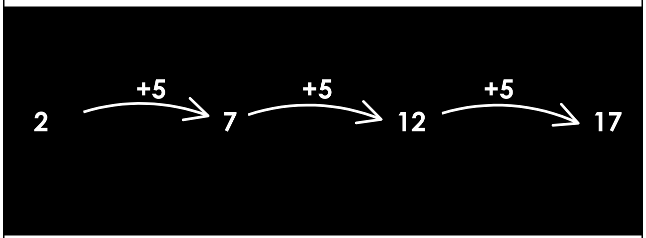
The sequence is then represented through a graph [29]:

→

29. *It is understood that the sequence starts from 2 again.*

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→



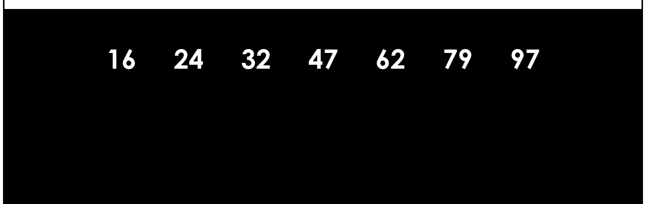
In these initial phases of the activity we mainly worked on the progression's 'step' and aspects related to the first number of the progression remained hidden.
As we will see in **Diary 25**, it is the time to get to point out the importance of defining the initial number of the progression.

Diary 25 (5th grade, October)

We work on the sequence

2; 5; 7; 12; 15; 17; …

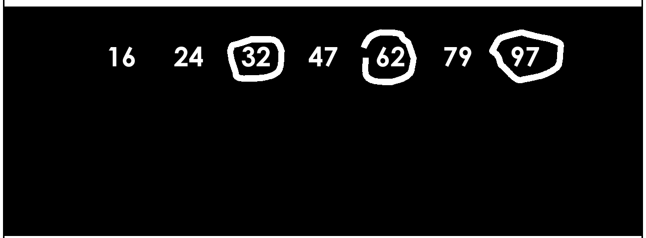
We seek characteristics of this sequence so that we can reproduce it.
√ «Which characteristics do we need to tell a classmate to enable him to identify the sequence?»
After some attempts the class decides that in order to identify the sequence they need to communicate:
(a) the starting number, in this case 2,
(b) the step, in this case +5.
√ The teacher writes some numbers on the blackboard and asks whether they belong to the sequence or not.



→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
The answer is generally quick: only numbers ending by 2 and by 7 belong to the sequence. The numbers are highlighted:



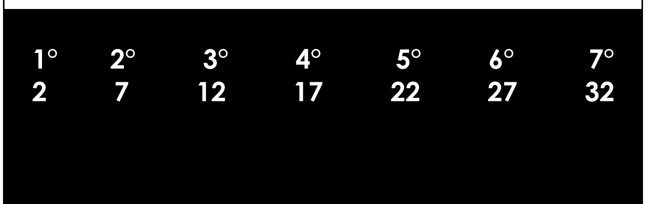
√ How many numbers are there in the sequence?
General and immediate answer: there are infinite, they never end.
√ Two types of activities are proposed straight after:
(a) write a sequence with the same rule but changing the starting number;
(b) invent sequences in which classmates will have to identify the rule.

10. Guessing the number

We carry on towards generalisation by proposing situations that provoke reflections on the relationship between *number* and *place* occupied.

Diary 26 (4th grade, November)

We make pupils reflect on relationships linking place in the sequence and number which is there.
Having noticed an initial difficulty, we prefer to mark places of the first numbers:



→

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→
√ «Any information about the 65^{th} number? And about the 124^{th} And about the 3571^{st} »
«The 65^{th} number will have 2 because the 65^{th} is odd»
«It will surely be with hundreds»
Some pupils are puzzled by this remark.
√ «Would you explain how you can recognise a number which is in a certain place?»
«If you look a the last figure of that place, if it is even we know that the number ends by 7, if it is odd we know that the number ends by 2»
√ She proposes a refinement of the law's definition.
«If the place number is even then the number ends by 7, if the number is odd, the number ends by 2»
«If the place is even the corresponding number ends by 7, if it is odd the corresponding number ends by 2»
√ «So, what number will be at the 20^{th} place?»
«The corresponding number ends by 7»
They cannot enact strategies that lead to the identification of the number.
√ They are asked how they can move from the 10^{th} (number 47) to the 20^{th} place. They all start making mental calculations.
A pupil doubles 47 and proposes 94.
They claim that it is not possible that a number of the series ends by 4.
«If the place is even the corresponding number ends by 7, if it is odd the corresponding number ends by 2»
√ She proposes that the sequence is continued beyond 47.
The sequence is constructed with the contribution of many pupils:

12°	**13°**	**14°**	**15°**	**16°**	**17°**	**18°**	**19°**	**20°**
57	**62**	**67**	**72**	**77**	**82**	**87**	**92**	**97**

«97!»

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
√ «And what number will be at the 30^{th} place?»
Pupils look focused. They think in silence.
√ «Think: how many times have wee added 10 to move from 47 to 97?»
«50!» **30**
√ «So: what number is at the 30^{th} place?»
«147!»
√ «Which at the 40^{th} place?»
«197!»
√ «Which at the 50^{th} »
«207, because it is enough to add 5 ten times»
√ «Which at the 60^{th} »
«257!!!» **31**

__30.__ The reasoning is carried out correctly but the answer should have been '5' (the number of times that 10 was added). It is possible that excited by the answer the class focused on the <u>product</u> (5) rather than on the process (10 × 5).

__31.__ Discussion presented in the Diary shows that pupils often have correct intuitions about the explored situations, but they cannot produce an argumentation.

In another class the same activity develops with different reasoning dynamics.

Diary 27 (5^{th} grade, October)

We try to understand which is the 10^{th} number of the sequence.
Pupils find 47 by trial and error.
We try to generalise.
√ The teacher asks the value of the 20^{th} number of the sequence.
The class is divided, and answers are chosen among the following :
«The 20^{th} number is 94 because if the 10^{th} is 47, the 20^{th} will be double, that is 94 ».
This hypothesis is refuted because 94 does not end by either 2 or 7 **32.**
«The 20^{th} number is 87 because add 10 steps, I must add 40»
The pupil cannot explain why 10 steps correspond to a shift of 40 units and not of 50 **33.**
They find by trial and error that the 20^{th} number is 97. They understand that every 10 steps you add 50. All agree with this conclusion.

→

__32.__ Interestingly a multiplicative strategy is enacted.

__33.__ A possible hypothesis is that the pupil has lost control of the situation but has decided to contribute anyway without grasping the inconsistency of his reasoning with respect to the situation. He might have related 20 and 5, that is the sought place number and the sequence step, and have got 4 as quotient.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
At this stage attempts of generalising fail, since pupils focus on the fact that going by 10 terms you get an increase of 50 and every 20 steps the hundreds' figure changes. **34**

11. From progression to its generating law

When the class is ready for the next step, they are led towards the formula that describes the sequence (2, 7, 12, 17, 22…)

Diary 28 (5th grade, October) **35**

√ «Which do you think will be the 20th number of the sequence?»
Paolo «It is 97, because I had written the sequence earlier and now I counted»
Ingrid «I did mentally (*but she cannot explain how*) and I found 97»
Giuseppe: «It is 102, because I did 5 times 20 plus 2, because I make 20 jumps of length 5 and I start from 2, sop I must add it» **36**
Alex «I get 97, because 5 times 20 minus 3 gives 97». He cannot justify his calculation, and he seems convinced that he was wrong and believes that Giuseppe is right.
All pupils verify that in the sequence constructed by Paolo, the 20th term is actually 97. Alex seems to have found the right formula, but …
√ «Let's seek the 30th number then»
Alex «It is 152» *(following Giuseppe's strategy)*
Paolo, who wrote the first 30 terms of the sequence, checks that it is true: he gets 147.
Pupils look daunted.
Giuseppe's explanation is convincing but it seems that Alex has proposed the correct formula.

→

34. *Diaries **26** and **27** show how interesting intuitions may become misleading if not guided and lead the exploration to a dead end.*
Often these risks intimidate teachers and limit their willingness to carry out explorations and discussions in a mathematical environment. These are precious 'tolls' to be paid to the understanding of how stimulating for both pupils and teachers could be to get involved.
On the methodological plane the teacher refines his own abilities to interpret the phases of discussion and to select the most productive interventions to reach the final mathematical objective. All this should be done respecting contributions often psychologically important (pupils who intervene very little, with a scarce self-esteem, with a limited linguistic baggage and so on) but scarcely or not at all meaningful in the achievement of a result.

35. *This diary shows an effective exchange on the dialectical plane, because pupils continuously refer to previous statements quoting the author. For this reason we preferred to keep the names of the participants. The teacher orchestrates pupils' exchanges discreetly but effectively.*

36. *Giuseppe does not control the sense of his calculation there are 20 places but the steps taken are 19.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

Some propose that the two strategies may coexist, and that they might use one or the other depending on the fact they are seeking an even or an odd term.

They decide to check and seek the value of the 25th number with Giuseppe's formula and with Alex's formula:

Giuseppe: $5 \times 25 + 2 = 127$

Alex $5 \times 25 - 3 = 122$

Paolo, in charge of checking, finds out that the 25th number is 122.

They agree on accepting Alex's formula. **37**

√ The teacher suggests that Alex's formula might be used to write some terms of the sequence.

Pupils find it easier to think of places that mainly correspond to tens.

Proposals are written on the blackboard:

20° = 5 × 20 - 3
25° = 5 × 25 - 3
30° = 5 × 30 - 3
100° = 5 × 100 – 3

√ «How can we work in the case of a bigger number? How can we represent the situation in the case of any number?»

Different proposals are formulated, more or less fancy, and they are transcribed :

(a) □ = 5 × □ - 3
(b) ★ = 5 × ★ - 3
(c) ….. = ….. × 5 - 3
(d) beep = 5 × beep – 3
(e) q = 5 × q – 3

The class chooses (e).

The author says that q stands for 'any number'. **38**

→

__37.__ The teacher tries to lead the class to generalise and formulate the law mathematically.

__38__. This is a great achievement although the successive coding is not clean. It is exactly in this alternation of ingenuous achievements, intuitions and linguistic inventions that <u>algebraic babbling</u> is constructed.

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→

√ The teacher remarks that the same symbol cannot be used with two different meanings. If q is for 'place number' it cannot also be for 'the number which is in that place'. The teacher tells the class that mathematicians invented a way to indicate 'the number which is in a certain place' and that Brioshi knows it too: n_q, a 'n' with a 'q' as underscored sign, which is read: 'the number which is in place q'. She writes on the blackboard:

$$n_q = 5 \times q - 3$$

√ The teacher asks if the formula includes information about the starting number, that is 2.

Pupils believe that this piece of information is not included in the formula.

It often happens that another class finds a different but equivalent formula. Next diary illustrates this possibility: it is proposed almost entirely because it is a good example of a consistent and well co-ordinated discussion. The sequence is the same, so that a comparison with previous diaries can be favoured.

Diary 29 (4th grade, November)

√ The teacher proposes the sequence:

2 7 12 17 22 27 32 …

→

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

→

√ «How can you explain how you find a number in the sequence?»

«I add 5, I add 5, I add 5, …»

√ «Translate it for Brioshi»

The pupil goes to the blackboard and adds:

2 7 12 17 22 27 32 …

2 + 5 + 5 + 5 + 5 + 5 + 5 = 32

√ «Must Brioshi do all these calculations?»

«We can write 5 times 6»

«No, it is 2 plus 5 times 6»

«It is always 2 plus 2 plus 2 and then plus 5 times 6»

√ «What if we wanted to find the number at the 20th place?»

«You do 2 + 5 × 19»

The class is puzzled; Lisa (the author of the last proposal) is invited to explain how she did.

Lisa «Because I tried at home and I multiplied by 20 but I saw it was not right because at the 20th place there was 97. So I took 1 out of 20 and I got it»

√ The teacher asks whether someone can explain why you must multiply by 20. They cannot answer.

√ They are asked to translate Lisa's last statement for Brioshi.

After some uncertainty, a proposal emerges and it is written on the blackboard:

2 7 12 17 22 27 32 …

2 + 5 + 5 + 5 + 5 + 5 + 5 = 32

2 + 5 × 20 – 1[39]

→

***39.** The formula highlights the importance of the class' early work since 1st grade on the different representations of a number 'around' its canonical form. This allows pupils to choose the most convenient representation in a certain situation, that is , as in this case, the one carrying the most meaningful information to proceed towards generalisation.*

As we have stressed many times, canonical form is often poor in meanings because it is synthetic and too obscure. In fact it only highlights aspects related to cardinality.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
√ She questions the correctness of this writing.
Pupils do not get the sense of her request; the teacher explains that they have just talked about parentheses.
√ «Imagine you send this formula to Brioshi. Do you guess how he would answer?» She writes: **40**

2 7 12 17 22 27 32 …

2 + 5 + 5 + 5 + 5 + 5 + 5 = 32

2 + 5 × 20 – 1

2 + 5 × (20 – 1)

Many pupils understand, probably because they find something known.
√ Pupils are asked to reflect on the first writing proposed by Lisa (2 + 5 × 6) and to explain it.
« 32 is at the 7th place and we added six numbers to 2, then at the 20th place it is 20 - 1 …» he leaves his reasoning incomplete.
√ They are invited to rewrite Lisa's first formula, rewriting 6 in another way.
After some uncertainty a pupil formulates a proposal which is written on the blackboard:

2 7 12 17 22 27 32 …

2 + 5 + 5 + 5 + 5 + 5 + 5 = 32

2 + 5 × 20 – 1

2 + 5 × (20 – 1)

2 + 5 × (7 – 1)

√ Pupils are asked to represent in the same way other bigger numbers. It is made clear that the important thing is not to find the number that is in a certain position, but rather to write how you can find it. **41**.

→

40. *Brioshi, is assigned the role of a sage pupil: he knows mathematics very well and uses its language correctly. Pupils must know this clearly and accept it as a 'rule of the game'.*
Of course Brioshi can make mistakes, but then he does it on purpose, in order to educate them, so to speak. For example, when pupils are stuck the teacher might write a wrong sentence at the blackboard and ask : "Excuse me, what would you say if Brioshi sent you this message?". Another possibility would be to 'provoke' pupils writing a sentence on the blackboard and asking them: "Do you think this answer by Brioshi is correct or rather there is something wrong? What do you think".
Concluding: Brioshi is a powerful mediator supposed that the class 'accomplices' both him and the teacher whenever he turns up.

41. *The relationship between 'solving' and 'representing' or between 'way to find a result' and 'rewriting the number in non canonical form' is a concept that needs a slow clarification over time. If pupils appropriate this concept and little by little learn to use it spontaneously, in grades 7th and 8th, when they deal with algebraic writings like $2x–3$ or a^2+b^2, positive outcomes of this work can be seen, because they become objects that carry meaning.*

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

→

Pupils show they got into the situation and the blackboard is quickly filled in with the following list:

2 + 5 × (7 – 1) = 32
2 + 5 × (20– 1) = 97
2 + 5 × (60 – 1) = 297

√ «Very good! And now a very difficult question! Looking at these numbers would you be able to represent the number which is in *any* place?»

Pupils cannot answer. 42

They are guided to reflect on the meaning of 'any'.

√ «Look, at the left hand side of equal. What changes? What stays the same?»

Pupils realise immediately that 7, 20 and 70 change.

√ In order to direct pupils' attempts they are told that mathematicians talk about 'first', 'second', 'third' and so on, and to indicate any number they talk about the 'n-th' number.

Some say they have already heard the expression 'for the n-th time'.

Little by little some proposals for Brioshi are made and written on the blackboard .

(a) 2 + 5 × (n° – 1)
(b) 2 + 5 × (e. – 1) 43
(c) 2 + 5 × (n – 1) 44

→

42. *Many times pupils propose random numbers; this cannot be avoided, especially with younger pupils because they do not have the concept of 'variable'. In this diary (as in others) we show how the obstacle can be avoided and the class be led to the intuition of this concept.*

43. *The author explains that 'e.' stands for 'n-th'.*

44. *This conclusion can be interestingly compared to that proposed by 5th grade pupils in Diary 26. In that case the initial 'perception' of the sequence had highlighted the 'rule' (+ 5) and shaded the first number of the sequence (2).*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
The last proposal is viewed as the best one because in previous writings numbers were not written in the ordinal (as in (a)), but rather in the cardinal form.
√ A new sequence is proposed and pupils are asked to find the new rule:

3 9 15 21 27 …

The rule is found very quickly:

$$3 + 6 \times (n - 1)$$

Since pupils show enthusiasm for their discovery, a final sequence is proposed and pupils are requested to write the new rule:

6 18 30 42 …

They write the rule very quickly:

$$6 + 12 \times (n - 1)$$

The meeting ends with a reflection on elements that characterise a sequence.

Next Diary continues Diary 28, about one month later.

Diary 30 (5th grade, November)

√ As you might remember, we wondered if, once we sent the formula characterising the sequence, Brioshi can understand that the starting number is 2 **45**.
The sentence we sent out is:

$$n_q = (5 \times q) - 3$$

Pupils claim that Brioshi does not have a clue.
A pupil hypothesises that Brioshi can understand that the first number is 2 «because 5 minus 3 is 2», but he cannot justify his intuition.
√ The class is invited to rewrite some numbers from the sequence on the blackboard, applying the formula again.

→

45. *The sequence the formula refers to is 2, 7, 12, 17, 22, … .*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

Posto: 1 2 3 4 5 6 7 …
Numero: 2 7 12 17 22 27 32 …

1° = (5 × 1) – 3
2° = (5 × 2) – 3
3° = (5 × 3) – 3
6° = (5 × 6) – 3
7° = (5 × 7) – 3

√ Pupils are invited to read formulae out loud **46**. The discussion is characterised by some attempts to read formulae 'literally' …

«The first number equals 5 times 1 minus 3»

… and more elaborated ones:

«To find the sixth number I do 6 times 5 and I take out 3»

«I applied the formula to the first number and I found 2 because 5 times one is 5 and minus 3 is 2!»

The class accepts the classmate's explanation, which focused, made explicit and justified what was a shapeless intuition for most of them. **47**

The initial sequence is proposed together with two additional ones:

2	**7**	**12**	**17**	**22**	**27**
8	**13**	**18**	**23**	**28**	
24	**29**	**34**	**39**	**44**	

The request is to identify the rule; the class easily recognises the rule '+5'.

√ «Is it enough to know the rule in order to characterise a sequence?»

→

46. *Continuous shifts between natural and mathematical language constitute a precious instrument in constructing the meaning of algebraic writings. Moreover it represents one of the most powerful educational instruments for the elaboration of algebraic babbling, since 1st grade.*

47. *This discussion, like others, reinforces the hypothesis that this type of activity contributes substantially to the construction of the initial concept of variable through a progressive stimulation of exploration towards generalisation.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→
«We need to know that final figures alternate»
«We must say the initial number»
Some examples clarify that to define a sequence two items of information are enough and these are fixed on the blackboard :

2	**7**	**12**	**17**	**22**	**27**
8	**13**	**18**	**23**	**28**	
24	**29**	**34**	**39**	**44**	

(a) The starting number
(b) the rule

The three sequences are analysed and the conclusion is that they have the same rule but different starting numbers.

Diary 31 illustrates through a clear and elaborated discussion how exploring the situation, with the continuous support of verbalisation, leads to achieving the formula. Strategies adopted by the teacher will be highlighted in Comments.

Diary 31 (5th grade, December)

√ A sequence is proposed. Pupils are asked to write down the way to identify its 37th term for Brioshi:

place:	**1**	**2**	**3**	**4**	**5**	**…**
number:	**3**	**11**	**19**	**27**	**35**	**…**

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
Pupils process their answer, that are progressively written on the blackboard and questioned.
«It is enough to calculate 37 – 1, therefore it is 36»
«No, that's not possible, can't you see that the 5th term is 35, how can 36 be the 37th term?»
«Sure, 37 must be multiplied by 8, which is the step, so» and writes:

37 × 8 - 1

√ «Hold on… not to waste time in calculations let us apply the formula proposed by your classmate to an easily checkable number: for instance that in the 8th place.
«8 times 8 is64 and minus 1 is 63»
Verification leads pupils to find out that the formula does not work, with disappointment of the author; in fact, if we continue the sequence written on the blackboard we get:

place:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**…**
number:	**3**	**11**	**19**	**27**	**35**	**43**	**51**	**<u>59</u>**	**…**

«It is no good! With our rule it is 7 times 8 that is 56 and minus 1 is 55, and not 59!»
√ The teacher asks which pupils to repeat which elements characterise a sequence.
«The starting number and the step»
√ **T**he teacher suggests they might write to Brioshi how you find the 4th term of a sequence having the following characteristics:

start 5 step '+9'.

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→

Some pupils, helping one another, add on the blackboard:

start 5 step '+9'.

5 + 9 × 4 = 5 + 36 = 41

We verify the proposal by transcribing the initial four terms of the sequence proposed by the teacher:

start 5 step'+9'.

5 + 9 × 4 = 5 + 36 = 41

1	**2**	**3**	**4**
5	**14**	**23**	**32**

«Then what we found is not right!»

«Yes, because the number of steps is 3 and not 4!»

√ The teacher tries to make pupils express the remark in a way that helps them to find the general formula «try and link together the number of steps and the number of the place» The class is guided to observe the sequence written on the blackboard.

start 5 step '+9'.

5 + 9 × 4 = 5 + 36 = 41

1	**2**	**3**	**4**
5	**14**	**23**	**32**

«The number of steps is the place minus 1!»

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→
The remark is highlighted with arrows.

start 5 step '+9'.

$5 + 9 \times 4 = 5 + 36 = 41$

1	**2**	**3**	**4**
5	**14**	**23**	**32**

√ Pupils are asked to describe in Italian how they find a term of the sequence. The following description is reached with several contributions:
«The starting number plus the step multiplied by the place number minus 1»
√ The definition is then translated in mathematical language for some particular cases (the initial uncertainty about parentheses is immediately sorted out):

start 5 step '+9'.

$5 + 9 \times 4 = 5 + 36 = 41$

1	**2**	**3**	**4**
5	**14**	**23**	**32**

10th term: **$5 + 8 \times (10 - 1)$**
27th term: **$5 + 8 \times (27 - 1)$**

√ The 'q-th' term is proposed then; the answer is easily given and written on the blackboard:

start 5 step '+9'.

$5 + 9 \times 4 = 5 + 36 = 41$

1	**2**	**3**	**4**
5	**14**	**23**	**32**

10th term: **$5 + 8 \times (10 - 1)$**
27th term: **$5 + 8 \times (27 - 1)$**

$q + 8 \times (q - 1)$

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

Next diary – the last for Situation 11– is almost a linking thread with Situation 12: it is not only about representing a sequence through a formula but also about interpreting the formula to understand which sequence it synthesises.

Diary 32 (5^{th} grade, December)

The class had sent a message to Brioshi, with the recorded sequence:

2, 7, 12, 17, 22, 27, …

Brioshi sends back a reply completely written in Japanese, in which the following formula emerges:

$$m° = 5 \times m - 3$$

The class is to understand the meaning of Brioshi's formula. The prevalent feeling is that the formula may describe the sequence the class sent to him. Pupils make some trials to verify their idea. This is an important step: the validity of a formula is tested.
The teacher recalls the fact that they had found out that to characterise a sequence, step and first term must be identified.
√ «So: do you think this formula contains information about the first term of the sequence?»
Pupils are puzzled, but most of them claim that the formula does not have that information and that the sequence may start from any number, provided that the step, i.e. 5, is respected.
A pupil gets out of the impasse claiming that the first number can be drawn from the formula. She goes to the blackboard and writes:

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→

m° = 5 × m - 3

1° = (5 × 1) - 3 = 2

Everything becomes clear.
«So we can see it, it is written! The first number is 2!»

12. From the rule to the sequence

The successive step concerns inserting the inverse operation with respect to those carried out so far: after starting from the sequence and finding the formula, we will try and reconstruct the sequence starting from the formula.
The support we used is a message by Brioshi, written in Japanese characters with the help of a Nippologist, containing a formula. This has a remarkable psychological impact on the class: Brioshi *exists!* **48**

48. *The message the teacher brought to the class is the following:*

ベッルーノの親愛なる友達へ
この問題に答えてください。

m° = 4 × m - 1 ?

君たちの友達より!

Brioshi

It goes without saying that the class is simply amazed.

Diary 33 (5th grade, December)

The message from Brioshi carries this rule:

q° = (4 × q) - 1

Pupils work individually. At the end of this phase many find the same solution, which is written on the blackboard:

q° = (4 × q) - 1

3 7 11 15 19 23 …

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

→
√ The teacher requires a justification of the followed strategy.
«I found the first number, 4 times 1 is 4, then I took out 1, and so the first number is 3»
Classmates agree
√ «How did you carry on?»
« I saw that you added 4»
Some pupils do not follow.
«Why do you add 4?»
The first pupil cannot make it clear, another one intervenes.
«I thought: in a formula we saw the other time there was 5 times q and we added 5, now there is 4 times q and so I thought we had to add 4»
Classmates agree, except Luca who asks, not being convinced: «Why do we need to add 4? If there s 4 times q, I think we need to multiply by 4»
«No, because 4 is the step you make»
Sara explains a more complex strategy.
«I found the first number like all others. Then, to find the second, I used the formula. $2^{nd} = (4 \times 2) - 1$, that is 7. At this point I found hat the sequence starter with 3 and carried on with 7. I thought I had to do always + 4. The third number would be 11. I checked with the formula and I got: $3^{rd}= 4 \times 3 - 1 =11$. Then I understood I did it right »
Pupils agree with Sara's reasoning.
√ The teacher asks to use the formula to find the values of the 20^{th}, 30^{th}, 100^{th}, and 1000^{th} numbers.
There are no difficulties. It seems that managing expressions with a variable is a well known practice.

Next diary offers a good example of argumentation used to illustrate a type of reasoning that differs from classmates' .

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

Diary 34 (5th grade, November)

√ One of the first messages by Brioshi written in Japanese is being presented. It contains the following formula:

$$m^{th} = 4 \times m - 1$$? [31]

The message meaning appears clear immediately: Brioshi proposes a problem himself: to guess a sequence starting from a formula.
The class starts working. Two different approaches are highlighted among those who understand how to do:
(i) most pupils calculate the 1st, the 2nd, the 3rd number in the sequence, and so on, applying the formula correctly.
(ii) Nicola uses a different strategy, verbalising it in a very effective way:
«I found the fifth number, calculating 4 × 5 –1 = 19. I then noticed that the new formula looked like the old one, and I thought that 5 in the first formula *(m's coefficient)* represented the sequence' step. Therefore in the new sequence the step should be 4, because the mysterious number m is to be multiplied by 4. So I know that the fifth number is 19, because I calculated it and I also know that I need to make steps of 4. But then I can write the sequence adding 4 to go ahead and taking out 4 to go backwards. ».

Previous diary illustrates a very common situation in classes undertaking these and other activities within an early approach to algebraic thinking. Two strategies may emerge when pupils face the interpretation of a formula representing a sequence.
One of these, extremely predictable, consists of applying the formula itself to particular cases chosen by the pupil, so that some terms of the sequence can be found.

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

The other one, more unexpected, shows a general capacity to work *by analogy*, linking two formulae – one related to known experiences, and the new one- and analyse them in parallel. In this second case pupils might not feel the need to find values because they are sure about the process, and operations' results become less important (as it actually happens in the case of Nicola in Diary 34).

We present now a diary of a very peculiar situation occurred in a 5th grade engaged with this activity. A pupil unexpectedly discovers a new strategy: one can work with *remainder classes*. Developments might be interesting and link to activities with grids of numbers, for instance. **49**

49. *V. Navarra G., Giacomin A.,* Ricerca di regolarità, la griglia dei numeri, *Pitagora Editrice, Bologna, 2003.*

Diary 35 (5th grade, November)

The class is reflecting on a message by Brioshi and on the possible translation of the formula

$$m^{th} = 4 \times m - 1$$

Pupils immediately remember how they decoded the problem, working by substitution and analogy. Unexpectedly an original approach emerges, completely different from others.

«I found the 5th number, substituting 5 for m. I did: 5°= (4 × 5) – 1 = 19. At this point I did: 19 : 4 = 4 with remainder 3. I did that because the step is 4, I understood that comparing the two formulae. If the remainder is 3, that means that the first number in the sequence is 3».

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

13. Reconstructing a hidden sequence

The path is enriched with a new difficulty. The teacher proposes a sequence with a missing number, to be discovered:

5 9 ….. 17 21 …

Diaries 36, 37 and 38 illustrate some examples related to possible attitudes in a class. Of course the reader must take into account the fact that the 4th and the two 5th grades the diaries refer to have acquired a notable experience concerning themes dealt with by ArAl project.

Diary 36 (4th grade, November)

√ The following sequence with a missing term is proposed and the task for pupils is to try and find it:

5 9 ….. 17 21 …

√ «Explain how you can recognise the number between 9 and 17 »
Pupils work individually. Protocols enable a classification of the followed strategies in three groups:
1) they start from the point of view of subtraction:
(a) "I sought the difference between 5 and 9"
(b) "To find the rule I did a subtraction which is 9 – 5 and I got 4 which is a rule"
2) they start from the point of view of addition:
(c) "Between a number and its successive in this sequence the rule is +4"
(d) I found out the rule looking at numbers 17 and 21, so I found out that the rule is +4. The operation I did was an addition.
(e) I did 9 + 4 = 13 and then doing 13 + 4 I got 17.
(f) I did 5 + 4 and it was 9 and then I understood that the rule was +4.
(g) I discovered the rule +4 by adding 4.
3) they give generic explanations:
→

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→
(h) I discovered the rule by observing the rule between 5 and 9.
(i) I discovered the rule by counting how many numbers are between one number and the other.

Diary 37 (5th grade, November)

√ «Explain how you could recognise the number between 9 and 17»
Pupils propose their own explanations and we write them on the blackboard.

9 + 4
17 – 4

💬 «I found half between 17 and 9»
√ The teacher asks that the last explanation be translated in mathematical language on the blackboard.
💬 «17 - 9 : 2»
√ «Are you sure that is correct?»
💬 «(17 – 9) : 2»

9 + 4
17 – 4

(17 – 9) : 2

In another 5th grade, after correcting some wrong proposals, linguistic aspects are refined and they try to get to a definition of 'sequence' (we remind here that the sequence is always the same: 5, 9, ….., 17, 21, … with a missing term to be identified).

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

Diary 38 (5th grade, November)

A pupil proposes to insert 14 (he added + 5 as in the first sequence)
A pupil proposes 15 (she summed up 5 + 9 + 1)
A pupil proposes 8 (difference between 17 and 9)
√ «Listen, but what do you think a sequence is?»
«The sequence is calculating»
«The sequence is that between each number there is always a plus »
«The sequence is that to get …» (he stops)
«The sequence is that you always need to add a number »
«The sequence is that you need to subtract»
√ A sentence is written on the blackboard, joining pupils' proposals and it is improved through successive changes.

A sequence is a list of numbers between which there is the same number

√ We write on the blackboard a list of numbers following the proposed description:

A sequence is a list of numbers between which there is the same number

1 3 5 3 9 3 13 …

Pupils realise that saying 'between numbers there is the same number' is no good.
The new, very excited discussion, leads to the following definition:

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→

A sequence is a list of numbers between which there is the same number

1 3 5 3 9 3 13 …

A sequence is a list of numbers between which the same figure is added

Pupils discuss on the difference between 'figure' e 'number'. New discussion and new collective definition.

A sequence is a list of numbers between which there is the same number

1 3 5 3 9 3 13 …

A sequence is a list of numbers between which the same figure is added

A sequence is a list of numbers between which the same number is added

Following many analogous interventions, the definition is slightly modified:

A sequence is a list of numbers between which there is the same number

1 3 5 3 9 3 13 …

A sequence is a list of numbers between which the same figure is added

A sequence is a list of numbers between which the same number is added

A sequence is a list of numbers between which one same number is added or subtracted

The last definition is accepted by the class. **50**

50. *The definition is accepted within the principles underlying the whole activity, that is algebraic babbling. It represents an achievement coming from a substantial collective effort. Nevertheless it is a 'dirty' and incomplete definition because the class did not consider the importance of the first term. It is up to the teacher to orchestrate another discussion aimed at merging the definition with its 'logical cleanness'.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Comments

14. Identifying progressions

A new problem situation is proposed: to find one or more sequences containing two given numbers. Two numbers are assigned in their natural order: given the nature of the generating law, the smaller will precede the bigger in the progression.
In the next four diaries given numbers are 7 and 19.
At the very beginning the problem seems very simple because the first sequence is immediately found by finding the difference between two numbers and being sure that the search is complete.
But it is exactly at this point that the most interesting part of the activity starts. Usually pupils find out – stimulated by the teacher - that arithmetic sequences are more than one.

Diary 39 (4th grade, November)

The teacher had assigned some homework: to identify some sequences containing 7 and 19; pupils dictate the series they found.
The following sequences are transcribed on the blackboard:

$7 \xrightarrow{+3} 10 \quad 13 \quad 16 \quad 19 \quad \ldots$

$7 \xrightarrow{+4} 11 \quad 15 \quad 19 \quad \ldots$

$7 \xrightarrow{+2} 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad \ldots$

$7 \xrightarrow{+6} 13 \quad 19 \quad \ldots$

$7 \xrightarrow{+12} 19 \quad \ldots$

$7 \xrightarrow{+1} 8 \quad 9 \quad 10 \quad \ldots \quad 18 \quad 19 \quad \ldots$

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Comments**

→

A pupil proposes a sequence that differs to all others:

$7 \xrightarrow{+20} 27 \xrightarrow{-8} 19$

We ask the class to continue it:

$7 \xrightarrow{+20} 27 \xrightarrow{-8} 19$

7 27 19 39 31 51 43

√ «Which is the common features to previous sequences? In what does this one differ?»

«You do an addition and a subtraction»

Pupils gradually highlight that the sequence proposed by their classmate is different from others because the rule is different.

√ We go back to the initial sequences and ask them if there are other possible ones.

The answer, in the negative, is not convinced; some wrong attempts arrive.

√ To help the class teachers suggest that the laws generating the first sequences be highlighted.

We write on the blackboard:

+3 + 4 + 2 + 6 + 12 +1 [51]

√ Reflection on these numbers is encouraged; the teacher points out that the concept of 'divisor' has not been dealt with.

→

51. *The phase starting now is meant to help the class to 'see' divisors of 12 in the 'steps', i.e. in the numbers of operators that generate the laws they have found.*
Let us analyse the situation in detail:
1) I observe that between 7 and 19 there is a difference of 12; so +12 is certainly a 'step';
2) I can also think that 7 and 19 are not subsequent numbers and 'break down' their difference in two equal parts; the new step in this case is +6, I get to find between them in the sequence number 13.
Going along this line of thought I find the other possible 'steps': +1, +2, +3 and +4.
Concluding: possible 'steps' are as many as the divisors of 12. In general, we might get to a collective construction – easier in a 6th grade than in a 5th grade- of a general definition of the following type:
"Given two numbers, the sequences that contain them and which start with the first of them are all those sequences having the divisors of the difference between the two numbers as 'step'".

Activities suitable for classes	*1*	*2*	*3*	***4***	***5***	***1***	*2*	*3*	**Comments**

→
√ A sequence with 5 and 15 is proposed.
The following proposals are shortly obtained:

5 $\xrightarrow{+10}$ 15 …

5 $\xrightarrow{+5}$ 10 15 …

5 $\xrightarrow{+1}$ 6 7 … 14 15 …

5 $\xrightarrow{+2}$ 7 9 11 13 15 …

√ Pupils are asked whether there are other possible sequences.
Their answer is no.
Reflection on both this and the previous sequence is protracted.
Operators are written on the blackboard again:

+3 + 4 + 2 + 6 + 12 +1

+10 + 5 + 1 + 2

«In the first one there are numbers and their halves » (we understand that the pupil refers to pairs 6 and 3, 12 and 6, 4 and 2)
«It is not half because there is no half 3 »
«Also 4 is not half 12»
√ Another sequence is proposed; it must contain 4 and 12. We get the following proposals:

4 $\xrightarrow{+8}$ 12 …

4 $\xrightarrow{+1}$ 5 6 7 … 11 12 …

4 $\xrightarrow{+2}$ 6 12 …

4 $\xrightarrow{+4}$ 8 12 …

→

Activities suitable for classes	1	2	3	4	5	1	2	3	Comments

→
√ The activity pleases the whole class. In trying to accumulate elements for reflection one last sequence is proposed: it must contain 2 and 17. The following sequences are discovered:

$2 \xrightarrow{+15} 17$ …

$2 \xrightarrow{+5} 7$ 12 17 …

$2 \xrightarrow{+3} 5$ 8 11 14 …

$2 \xrightarrow{+1} 3$ 4 5 … 17 …

Many pupils seem to have intuitions.
√ The teacher suggests them to rewrite the 'rules numbers' in order:

1) contains 7 and 19:	**12**	**6**	**4**	**3**	**2**	**1**
2) contains 5 and 15:	**10**	**5**	**2**	**1**		
3) contains 4 and 12:	**8**	**4**	**2**	**1**		
4) contains 2 and 17:	**15**	**5**	**3**	**1**		

«There is always 1»
Pupils do not 'see' anything else. **52**

The problem "Find out sequences containing two given numbers" leads to deal with the issue of divisors of a number, as we explained in Comment 33. Next diary deals with this situation through a variant of the 'bus metaphor' (see Diary 7).

52. *We link back to previous comment. Probably 4th grade pupils are not familiar with the instruments they would need to 'see'. For example, they have weak knowledge about divisors; and most of all weak* dynamic *knowledge about divisors.*
With older pupils a deeper analysis of the links between initial steps of the four (12, 10, 8, 15) and the numbers at their right on the blackboard will be possible; moreover they might search for common features (numbers at the right are divisors of the related steps).

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

Diary 40 (4th grade, December)

√ We ask which elements enable identification of an arithmetic progression. 53

A short discussion leads pupils to the conclusion, already met elsewhere, that starting number and step are enough.

√ The following problem is thus proposed: to discover how many sequences containing 9 and 17 can be found.

Pupils start working individually. Soon 'classic' questions arise:

«Can we write some numbers before 9?»

«Can we invent as many sequences as we like? 54»

After a while pupils start proposing their own solutions (the first number represents the first number of the sequence, the second number the step) and these are progressively written on the blackboard:

(1)	5	+ 4
(2)	9	+ 2
(3)	4	+ 1
(4)	9	+ 8

√ The teacher suggests that the situation be <u>represented</u> through a table:

(1)	5	+ 4
(2)	9	+ 2
(3)	4	+ 1
(4)	9	+ 8

1° termine	passo
5	+ 4
9	+ 2
4	+ 1
9	+ 8

→

53. *Si ricorda che in matematica quella che le classi hanno chiamato sinora 'passo' si preferisce chiamarla 'ragione'.*

54. *These questions hide some important problems characterising mathematical argumentation. Pupils act instinctively and do whatever they can, show up their knowledge, do as much as they can being afraid they are not doing enough, and so on. The issue would deserve deepening in another context.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→
√ The teacher tries to make pupil think about the steps they have identified. They are asked whether other sequences could possibly be found and it is only for their incapacity that they have not been found. Some support the latter hypothesis.
√ A variant is proposed: the sequence must now contain 2 and 20.
Pupils work individually; after a while the following writings appear on the blackboard:

	step
2 ⟶ 4	+ 2
2 ⟶ 8	+ 6
2 ⟶ 3	+ 1
2 ⟶ 5	+ 3
2 ⟶ 20	+ 18
2 ⟶ 11	+ 9

All sequences have been identified and representation is more synthetic; the divisors of 18 are not recognised as such.
√ In making another proposal (30 and 54) we decide to link intuition to a concrete situation. [55]
We draw a line, divide it into equal dashes and we start telling that it represents the bus route. [56]

Between stop number 30 and stop number 54 there are 24 intermediate stops.
Some buses, starting from number 30, stop at each stop, others every 2 stops, every 3, every 4, every 5 and so on.
→

__55.__ A metaphor already used in the activity of Diary 7 is recalled.

__56.__ This is not a simple process and requires a preparatory work on identifying the number of divisors of a number so that it might be easier to recognise them in situations like this one.
These activities may be successful depending on habits set up by the teacher in the class, especially in relation to the fact that pupils should get used to seeking links between data emerging from a problem situation.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→
Stop 54 is that of a school. Pupils who get there by bus should know which are the 'good' stops, that is those at which the buses going to 54 stop.
Pupils start imagining buses and their stops.
They understand that a first bus (that we name +1) stops every time and therefore it is right.
Another one (+2) stops at 30, then at 32, at 34 and so on up to 54.
A third one (+3) at 30, and then: 33, 36, 39, … and what about 54? Quick calculations lead to understand that this one is right too.
It is ok also that one (+4) that stops every 4 stops: 30, 34, 38, 42, 46, 50, 54.
We try that one (+5) that stops every 5 stops, ma we realise that it does not stop at 54: 30, 35, 40, 45, 50, 55.
Some say that you can always go back, but it is clear to all that this would contradict the 'rule of the game'.
Every time we find a bus that works we represent its route with coloured arcs.
Pupils discover in this way that buses +1, +2, +3, +4, +6, +8, +12, +24 all work and there are no others.

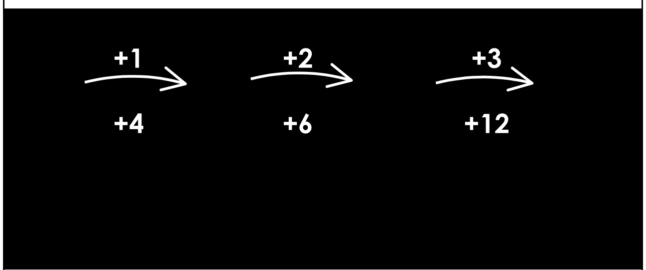
It might also happen that favourable environmental conditions make the class closer to identifying divisors, or at least to the an intuition of the direction toward a fruitful search.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

Diary 41 (5th grade, November)

√ A problem is posed: Marta wrote an arithmetic sequence in her workbook, but some spots dropped on it. Only two numbers are visible: 7 and 19.

Can you find the sequence that Marta wrote?
Pupils start working and they immediately find more than one solution.
√ We try to understand which solutions have been found and how.
Pupils choose a representation through graphs and identification through their steps.
We write on the blackboard:



They are asked how they found these solutions:
- I did the operations: 7 + 12 = 19, etcetera.
- But there is also subtraction and therefore if there is step +2 there must be also –2.
- If there were 'minus', number 19 would have come before 7, but it is the converse, hence in my opinion we must work with addition, no way.
- (Emmanuele) I tried all the times tables.

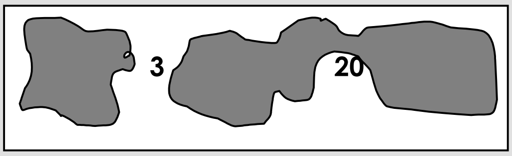
→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→
I did a regularities table: I started from 7, and I added 1, then 2, then 3 and so on, trying to see whether 19 was in the sequences ***57***.
I thought that from 12 to 19 I could make a jump of 12, but also two of 6, because 2 times 6 is 12… and also 3 times 4 is 12!
(Alberto) In this sequence you can only find steps of 2 or multiples of 2, or with numbers that either summed up or multiplied give 12 ***58***.
No, because also 5 added to 7 is 12, but it does not work: if I use step 5 starting from 7, 19 is skipped.
Only numbers in the times table that have 12 are right, and actually if we look at the solutions we found they are all the times tables containing 12.

Expansion 4: About 'problems with spots'

Almost all pupils from 5th grade onwards can understand the role played by divisors in the solution of problems like those we are exploring.
The teacher can take advantage of this situation to introduce, or recall, prime numbers.
The activity is globally suitable for junior high school.
Let us analyse the example of a problem with spots, in which the only visible terms of the progression are 3 and 20.



Analysis leads to find out that only two sequences contain simultaneously 3 and 20, because the difference between 3 and 20 is 17 and the only divisors of 17 are 1 and 17.

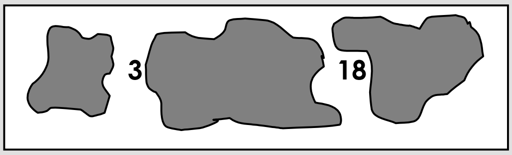
→

57. *It is important to remark that Emmanuele did not stop at the first solution, but looked for others. We want to underline the educational importance of situations like this one which are not viewed as 'school problems', thus contributing to weaken the stereotype according to which problems have always one and only one solution.*

58. *Alberto moves towards a more general view of the problem, despite linguistic inaccuracies. We might think that to him 'numbers that summed up give 12' means 'numbers that summed up more than once to themselves give 12'.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→
The main difficulties that can be met are linked to a loss of control of the situation's data. We clarify this through an example.
Suppose you have a sequence in which only 3 and 18 are visible:



If pupils are convinced that they must look for divisors, many of them will not probably take into account 3, and therefore they might seek only the divisors of 18, the bigger number.
Direct verifications that numbers 3 and 18 belong to progressions generated by divisors of 18 leads them to find some counter-examples.
For instance, going by 2s starting from 3 (initial element), they will only get odd numbers and therefore they will 'skip' 18:

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, …

At a more abstract level one could say that applying the operator +6 starting from 3, the numbers you gradually obtain are of the type

$$3 + 6k$$ **59**

(all have remainder 3 with respect to the division by 6). **60**
The problem becomes:

is18 reachable starting from 3
with 6-long steps?

This question can be translated in mathematical language (one might effectively say: for Brioshi):

$$18 = 3 + 6k$$

Since 18 = 3 + 15, we must see whether 15 can be a multiple of 6. In a more abstract way: see whether the equation

$$6k = 15$$

→

59. *The writing can be better understood going from particular cases toward generalisation:*

$3 \xrightarrow{\times 6} 3 + 6 \times 1$

$3 \xrightarrow{\times 6} \xrightarrow{\times 6} 3 + 6 \times 2$

$3 \xrightarrow{\times 6} \xrightarrow{\times 6} \xrightarrow{\times 6} 3 + 6 \times 3$

For any number of steps we will have:
3 + 6 × k
or an equivalent form:
3 + 6k

60. *The issue of remainders of the division will be clarified in the next* ***Expansion 5****.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→
has solution, that is, whether it exists a natural number that multiplied by 6 is equivalent to 15.
We can proceed in an analogous way assuming the other divisors of 18 as steps.
Step 2 leads to the equation 2k = 5 which does not have solution.
Neither the equation 9k = 15 has solution.
Step 3 (equal to the initial element) produces a progression containing both 3 and 18. As a matter of fact, the equation

$$3k=15$$

has solution k = 5, and this tells us that after five steps starting from 3, you get to 18.
We remark that differently to 6 and 9, 3 is a divisor of 15 (18 - 3, difference between the two given numbers). Therefore the role of this difference emerges together with that of possible divisors.
It is important that pupils reflect on the fact that the point is the *relational link* between two given numbers, which can be viewed as:
- One is a divisor of the other,
- They share common divisors,
- They are co-prime,
- ...

Each of these ways leads to a specific deeper analysis.

Diary 42 (5th grade, December)

√ The incomplete sequence containing numbers 7 and 19 is presented.
Pupils identify 6 solutions, corresponding to steps +1, +2, +3, +4, +6, +12.
√ They are asked how they found these sequences. Many show they have made trials, but there are also different interventions.
«Since the distance between 7 and 19 is 12, it is enough to make different steps up to 12»
«Yes, but steps of 5 do not work»
«Neither steps of 7 do»
→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→
√ A different incomplete sequence is proposed, wit visible values 3 and 18.
Most pupils proceed by trial and error (as their workbooks show), but some only write down the steps: they are steps of 1, 3, 5, 15. The teacher asks them how they did.
«I found the distance between 3 and 18, it was 15: then I tried to divide 15 in equal parts: I can do 3 jumps of 5 »
«Or rather 5 jumps of 3, or rather 1 jump of 15, or again 15 jumps of 1»
√ «Could you apply this same method with the previous incomplete sequence too?»
«Yes, because the distance between 7 and 19 is 12, and therefore to make 12 you do equal jumps, and so three jumps of 4»
«Or rather two jumps of 6»
«Or rather six jumps of 2»
«Also four jumps of 3, or twelve of 1 , or one of 12»
√ Other two incomplete sequences are proposed: in one of them values 5 and 13 are visible, in the other one 8 and 15.
Pupils work in a correct way, getting to solutions without difficulties.

Expansion 5: If the smallest between two numbers is not the first term of the progression

The problem situation is presented in these terms:

> *Identify all progressions that contain two given numbers not necessarily starting from the smaller one.* **61**

→

61. In quest'attività l'operazione di divisione gioca un ruolo di grande importanza e di notevole impegno concettuale per quanto concerne l'interpretazione del quoziente e del resto.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→
Pupils will be guided to discover that, for each possible step they will need to compare the first number with the step.
If this number is lower than the step the progression will start from there.

If the number is higher than the step, once the division has been made:
- the remainder will give the first term of progression containing the two numbers; **62**
- the quotient of this division will indicate the place number (after the first) where the first of the given numbers is located.

Some examples:

Example 1.

> Identify all progressions containing the pair of numbers 14, 25.

The difference between 25 and 14 – that is the step– is 11, and less than 14. Hence the only possible progression is that having 14 as first term:

<u>14</u>, <u>25</u>, 36, 47, 58, 69, …

Example 2.

> Identify all progressions containing the pair of numbers 14, 17.

The difference between 17 and 14 is 3, which is a prime number, hence the possible steps are 1 and 3.

(2a) with step 1: you get the natural progression, generated by operator +1, coinciding with the natural series:

1, 2, 3, 4; … <u>14</u>, 15, 16, <u>17</u>, 18, ….

→

62. *Per una visione unificante conviene considerare anche il caso in cui il resto sia 0.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→
(2b) with step 3: dividing 14 by 3 you get quotient 4 and remainder 2.
The progression generated starting from remainder 2 is then:

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, … **63**

From this progression you get others containing 14 and 17, respectively starting

with 5: 5, 8, 11, 14, 17, 20, 23, 26, 29, …

with 8: 8, 11, 14, 17, 20, 23, 26, 29, …

with 11: 11, 14, 17, 20, 23, 26, 29, …

with 14: 14, 17, 20, 23, 26, 29, …

The total number of progressions containing 14 and 17 is six.

Example 3.

> Identify all progressions containing the pair of numbers 14, 20.

The difference between 20 and 14 is 6, and hence possible steps are 1, 2, 3, 6.

3a) with step 1: you get the natural progression.

3b) with step 2: dividing 14 by 2 you have quotient 7 and remainder 0. **64**
With step 2 you get eight progressions:

with 0: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, …

with 2: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, …

with 4: 4, 6, 8, 10, 12, 14, 16, 18, 20, …

with 6: 6, 8, 10, 12, 14, 16, 18, 20, …

with 81: 8, 10, 12, 14, 16, 18, 20, …

with 10: 10, 12, 14, 16, 18, 20, …

→

63. *14 is at the 5^{th} place in the progression in which 2 is the first term.*

64. *14 is at the 7^{th} place in the progression in which 0 is the first term.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→

with 12: 12, 14, 16, 18, 20, …

with 14: 14, 16, 18, 20, …

3c) with step 3: you get, besides natural progression, the same five progressions of previous problem, respectively starting:

with 2: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, …

with 5: 5, 8, 11, 14, 17, 20, 23, 26, 29, …

with 8: 8, 11, 14, 17, 20, 23, 26, 29, …

with 11: 11, 14, 17, 20, 23, 26, 29, …

with 14: 14, 17, 20, 23, 26, 29, …

3d) with step 6: dividing 14 by 6 you get quotient 2 and remainder 2. [65]
With step 6 you have three progressions, respectively starting:

65. 14 is at the 2nd place in the progression in which 2 is the firs term.

with 2: 2, 8, 14, 20, 26, 32, 38, ...

with 8: 8, 14, 20, 26, 32, 38, ...

with 14: 14, 20, 26, 32, 38, ...

Hence progressions containing 14 and 20 are 17 altogether. [66]

66. In general, as we saw, for each progression starting with the first of the two numbers there are other two progressions that are parts of it.
You actually make a partition of the set of progressions containing the two terms, putting progressions with the same step in the same subset.

Exploring results of these explorations you can find that, for each step, the number of possible progressions equals the quotient of the related division plus 1.

15. Sequences and Brioshi

"What information is enough to identify a sequence?"
As we saw in previous diaries, pupils' answers to this question are manifold and their difficulty in constructing collective reasoning lead to a continuous search for strategies that favour understanding.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

The following diary – the last in this Unit – illustrates one of these strategies enacted in a 5th grade: this is an interesting anticipation of how an actual exchange of messages with Brioshi's class would be possible.

Diary 43 (5th grade, January)

√ The following arithmetic sequence is proposed:

2 7 12 17 22 …

and the information allowing pupils to identify it is required.

Answers lead only to the list of numbers or, at most, to the identification of the step (the little 'arc' with above the writing +4, for instance, is proposed).

√ In the attempt to make pupils understand that this information is necessary but not sufficient, the teacher invites pupils to propose solutions that can be obtained with step '+4'.

The following proposals are made:

6 10 14 …

4 8 12 …

5 9 13 …

√ The teacher points out that the step enables identification of several sequences, not only one.

Some pupils suggest that the first number of the sequence should be stated too.

This seems clear but when the teacher requests an example, the following proposal is made:

3 7 19 23 …

√ We and some pupils are amazed and ask for explanations.

The last proposal's author explains the sequence in the following way: the first number is 3, then there is step +4 and then step +12. He cannot explain the reason for this change. [67]

√ In order to understand the class' difficulties and be able to intervene, the teacher asks everybody to invent a sequence and then propose the information they would send to Brioshi to make him find out the sequence itself.

→

67. *The pupil might have taken into account the first term of the progression (3) and multiplied it by the step (4) thus getting 12. This would be the classic case in which students lose control of the process' meaning and go straight to the operation, since they do not know what to do.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→

A pupil proposes to write 12 + 5. **68**

Many detect ambiguities in the representation because it looks like an addition. They agree on separating 12 and +5 with a semicolon and the following change is made:

(a)12; +5 38

(b) $20 \xrightarrow{+19} 39 \xrightarrow{+19} 48 \xrightarrow{+19} 57$

$57 \xrightarrow{-19} 48 \xrightarrow{-19} 39 \xrightarrow{-19} 20$

(c) $16 \underset{-8}{\overset{+8}{\rightleftarrows}} 24 \underset{-8}{\overset{+8}{\rightleftarrows}} 32$

(d) $9 \xrightarrow{+4} \xrightarrow{+4}$

(e) $12; \xrightarrow{+5}$

(f) $12 \xrightarrow{+5} \xrightarrow{+5} \xrightarrow{+5} \xrightarrow{+5}$

(g) $7 \xrightarrow{+4} \dots \xrightarrow{+4} \dots$

(h) $4 \xrightarrow{+6} \xrightarrow{+6}$

1 2 3

Discussion goes on with some uncertainties.

A pupil suggests that (a) be excluded because he cannot understand it.

The author of (a) proposes that some dots are added.

She underlines that repetition of the step is not necessary.

Several pupils claim instead that it is necessary, otherwise Brioshi would not understand what he should do.

→

68. *This change will later block more than a pupil who did not grasp the sense of the convention agreed with the classmate.*
A possible reason for this lies in the fact that some moments of an exploration-discussion are extremely delicate. In fact, although they are not striking- like in this case- they nevertheless represent a subtle but important step forward to the achievement of a meaning. Getting detached from the discussion, although for few moments, interrupts control of the collective reasoning consistency, and the 'absent' pupil might be disoriented when it comes to understanding the successive moment.
The social construction of knowledge is fruitful because it trains pupils to keep focused.

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	**Commenti**

→
√ The class is reminded that Brioshi is working at the same topic and is preparing some problems for us; therefore he is facing the same difficulties and is able to interpret the messages we will send to him.
A pupil proposes to exclude (b) and (c) because they give Brioshi a started sequence and therefore there would be no problem. The class agree.
«I do not understand (h). If I sum 1, 2 and 3 I get 6 and it is not a number of the series»
The author of (h) explains that they are not the sequence's numbers but rather place numbers.
Pupils decide to eliminate (h) too.
√ Teachers repeat what has been just agreed: it is not necessary to repeat the step.
They propose that also (d), (f) and (g) be cancelled.
The pupil who had suggested they could consider only the first term and the step underlines her idea again.
At this point there are only the following proposals on the blackboard:

(a) 12; + 5
(e) 12; $\xrightarrow{+5}$

The class cannot decide **69**.
√ To overcome the impasse, we decide to face the exchange of messages with Brioshi more realistically. We decide to name 'B' Brioshi's class and 'N' 'us, 5th grade'. Pupils are asked to imagine they are in a multimedia room and write messages on the computer. **70** We know that the screen will show our messages and Brioshi's simultaneously, in real time. The class is thus invited to formulate answers that will be written on the blackboard as if it were the computer screen **71**.
→

69. *This part of the activity involves knowledge of the convention concept. Both writings (a) and (e) are acceptable because they carry the same meaning; pupils do not think so, because the first one is more abstract, whereas the second, with the arrow, is more 'concrete' and therefore more understandable.*

70. *The class has previously experienced an exchange of messages with a 'Brioshi class' through the use of the Netmeeting Messenger software.*

71. *As we will clearly see, the activity gradually becomes very realistic, favouring increasing identification and involvement by the class. The co-ordinator, who changes his role a number of times, will have to invent answers that will not only be made explicit in mathematical language, but will be also expressed more effectively with several symbols, so that stimulating 'emotions' for the class may be communicated.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→
B: 12; +5

The class is puzzled; finally these proposals appear on the blackboard and they must choose among them:

Na: 12 + 3 = 15
Nb: 22
Nc: 15 – 12 = 3

They do not know what to do.
√ The teacher tries to overcome the impasse by proposing a new message by Brioshi: this seems to help the class using the arrow symbol for the step (practically, proposal (e)):

B: $12; \xrightarrow{+5}$

The class answers:

N: 12 + 5 = 17

√ Brioshi's answer is immediate:

B: 12 + 5 = 17 ?!?!?!?!?
12; +5 !!!!!!!!!

The following exchange comes straight after:

N: 17 22 27 27
B: **12**; + 5 !!!
After some amazement the following proposal is finally made:

N: 12 17 22 27
B: OK

At the end of the exchange the following messages are written on the blackboard:

→

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→

B: 12; +5

Na: 12 + 3 = 15
Nb: 22
Nc: 15 – 12 = 3
??

B: 12; $\xrightarrow{+5}$

N: 12 + 5 = 17

B: 12 + 5 = 17 ?!?!?!?!?
12; +5 !!!!!!!!!

N: 17 22 27 27

B: 12; + 5 !!!

N: 12 17 22 27

B: OK

√ Teachers try to consolidate the concept of the two necessary items of information: the fact that the semicolon is a choice made by Brioshi is made clear and also another symbol can be used but then we do not know whether he will understand it. We propose to send a problem to Brioshi and ask the class to invent it.

N: 20 / + 6 **72**

√ A pupil is invited at the blackboard to write down the answer, imagining he is Brioshi.
He writes:

B: OK

Rumble in the class.
A pupil proposes:

B: 20 26 32 38

The class suggests they can write.

N: OK

→

72. *Accepting the rules is actually a complex issue at this age. As you see, although we have dealt with symbols to be used to separate the first number from the step, a completely different symbol, invented for that specific case, appears in the message and the class accepts it and will be using it straight after. The teacher is right in accepting it herself: she postpones the proposal of a socially shared code which would be imposed prematurely.*

Activities suitable for classes	1	2	3	**4**	**5**	**1**	2	3	Commenti

→

The exchange on the blackboard is the following:

N: 20 / + 6

B: OK

B: 20 26 32 38

N: OK

Exchanges of problems follow, with increasing control of the situation by the class.

N: 20 / + 10
B (Davide): 30 + 10 = 40
N (√): ??? [73]

√ The meaning of the two numbers is recalled.

B: 20 30 40 50
N: OK

The exchange on the blackboard:

N: 20 / + 10

B: 30 + 10 = 40

N: ???

B: 20 30 40 50

N: OK

√ They decide to formulate Brioshi's message differently, in order to get a new problem for the class to solve.

B: 5 11 17 23 …
After some hesitation the class answers:

N: +6
√ We try to increase the tension:

→

Commenti:

73. *The teacher interprets pupils' bewilderment following Davide's proposal through a reply by Brioshi.*

Activities suitable for classes	1	2	3	4	5	1	2	3	Commenti

→

B: ?

Almost immediately the class proposes satisfactorily:

N: 5 / +6

B: OK 74

The exchange on the blackboard:

B: 5 11 17 23 …

N: +6

B: ?

N: 5 / +6

B: OK

74. *The diary highlights the sometimes contradictory, inconsistent and debated path followed by the class. But it also shows the need for these creative phases. Too often in everyday teaching we forget that the knowing process is rarely linear and the strongest achievements are those constructed through bewilderment and suffering.*
Of course, as Newton would say, in all this the giant's shoulders are fundamental. And this is a relevant role for teachers.