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General aspects

1. The ArAl Project

The ArAl Project is meant to innovate the teaching of arithmetic and algebra in both primary and lower secondary school. The project is located within the *early algebra* theoretical framework, according to which *the main cognitive obstacles in learning algebra often arise in arithmetical contexts, in unexpected ways, and may later bring about conceptual obstacles – that may be insurmountable- to the development of algebraic thinking*.

A brief illustration of the main points of this hypothesis is needed here.

The international literature dealing with research on mathematics learning and in particular on the learning of algebra and on related difficulties – at different age levels, from the beginning up to university- highlights a widespread crisis of the traditional teaching of algebra. The identified reasons are very different in nature: *cognitive* reasons (algebra is difficult *per se*), *psychological* reasons (algebra intimidates), *social* reasons (the environment passes on *phobic* attitudes towards mathematics), *pedagogical* reasons (students seem to be less and less motivated towards studying especially when higher performances are requested), *didactical* reasons (stereotyped and inadequate methods).

Algebra, as language characterising a *higher* mathematics, represents a sort of wall for many students, mainly because they often have a weak conceptual control of *meanings* of both algebraic *objects* and *processes*. In the last twenty years research focused on a wide number of possible approaches to develop this type of control, for instance problem solving, functional approach, approach to generalisation.

Among other the *linguistic* approach is becoming increasingly important: it starts from a conception of *algebra as a language*. In this perspective the strong hypothesis of ArAl Project is that there is an *analogy* between ways of learning natural language and ways of learning algebraic language; the *babbling* metaphor can be useful to clarify this point of view.

Learning a language the child gradually appropriates its meanings and rules, developing them through imitation and adjustments up to school age when he will learn to read and reflect on *grammatical* and *syntactical* aspects of language. In the traditional teaching and learning of algebraic language the study of rules is generally privileged, as if formal manipulation could precede the understanding of meanings. The general tendency is to teach the syntax of algebra and leave its semantics behind. Mental models characterising algebraic thinking should rather be constructed within an arithmetical environment – starting from early years of primary school – through initial forms of *algebraic babbling*, teaching the child *how to think arithmetic algebraically*. In other words, algebraic thinking should be progressively constructed in the child as both an instrument and an object of thinking, strictly interweaved with arithmetic, starting from its meanings.

For this purpose it is necessary to construct an environment able to stimulate an autonomous elaboration of algebraic babbling and consequently to favour the experimental appropriation of a new language in which rules may be gradually located, within the constraints of a didactical contract that tolerates initial moments of syntactical 'promiscuity'.

General aspects

2. ArAl Units

The Units are an important result of ArAl Project and they are designed for a wide diffusion of the project itself; they can be viewed as *models of processes of arithmetic's teaching in an algebraic perspective* and are meant to provide teachers an opportunity to reflect on both their knowledge and their modus operandi in their classes before offering teaching paths to implement in class.

The 'fine tuning' of each Unit of ArAl project is the result of a process lasting at least three years, organised through a sequence of phases:

a) The choice of themes to be investigated

- At the beginning of each school year the themes around which experimental projects will be articulated are elaborated;

b) Experimental setting in the classes: joint lessons, minutes

- each project – launched by an extremely flexible sequence of problem situations- is developed throughout the year in several experimental classes; in primary school classes, teachers and teachers-researchers[1] simultaneously carry out the project through joint lessons;
- class teachers write minutes for every meeting (taking notes, making audio recordings or vide recordings in different situations) collecting a high amount of documental material (discussions, written protocols, methodological notes, unforeseen events, reflections, hints and so on);
- class minutes – which represent a fundamental instrument for the analysis of the teaching/learning process within the project- are transcribed into electronic form by class teachers and sent out to teachers-researchers who carried out the activities in a joint lesson;
- purposefully collected minutes are periodically spread to the group;
- in between two subsequent joint lessons class teachers clarify and deepen with their students some aspects which were left incomplete, propose reinforcing problems, collect meaningful materials.

c) Transition to the Units

- at the end of the school year minutes of each class are globally revisited on the basis of carried out discussions and organised in the form of embryo of a Unit to be tested later in classes participating in the project as well as in external classes.

[1] The role of teachers-researchers is an Italian peculiarity. In the early 70s, following innovative ideas brought forward during the 60s, spontaneous meetings between university and non-university teachers give rise to this figure. From these meetings Research Kernels in Mathematics Education are constituted, patronised by CNR and by CIIM (Italian Committee for Mathematics Teaching), a sub-organisation within UMI (Italian Mathematical Union).

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d) Writing up the Units in their final version

- when the collected elements are considered sufficient the Unit is organised in its final version through the elaboration of the most significant parts of collected minutes (which may be over one hundred in the case of compelling Units).
- The Units are structured so that they can:
  - Describe – in the left hand side of each page- a reasoned sequence of synthesised didactical paths carried out with constructive modalities,
  - Make transparent- in the right hand side- aspects, deduced from analytical reading of minutes (see previous point b), which can help the teacher in the implementation: methodological choices, enacted dynamics, key elements in processes, extensions, pupils' potential behaviour, difficulties and so on.

e) The Unit is published.

The Units are meant to be used in the classroom but their actual implementation requires a theoretical study. Two basic instruments of the project have been elaborated to this purpose: the *reference theoretical framework* and the *Glossary*.

Four sites host ArAl materials:
(1) www.aralweb.it
This is the official web-site, documenting the project in its scientific, methodological and educational aspects, concerning materials already published in the ArAl series as well as 'in progress' materials. Increasing space is given to the use of new technologies in mathematics education – from primary to lower secondary school- in an *e–learning* perspective.
(2) www.eun.org
Educational platform of the European Community which hosts the ArAl Community.
(3) www.matematica.unimo.it/0attività/formazione/grem
English version of the project; it is within the website of the Mathematics Department of the University of Modena and Reggio Emilia.
(4) www5.indire.it:8080/set/aral/aral.htm
The web-site is inside the Indire pages. It includes part of the materials of ArAl project, selected together with other 27 within the national contest SeT [2](2001) and funded by MPI.

[2] SeT: Special project for scientific and technological innovation

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3. The Glossary

Some terms that appear in each Unit constitute the **keywords** in the theoretical context of the Project. A correct understanding of these terms permits to set the proposed activities within a framework that is consistent with the inspiring principles as well as with other Units.

For this reason the Glossary can be viewed as the actual turning point for the whole ArAl project, in that it is constructed in order to promote and support, together with the Units, reflection by the teacher not only around themes developed in them, but , and more generally, on knowledge and convictions that lead him/her to explore delicate links through which the complex relationships linking arithmetic and algebra are made explicit.

The set of keywords elaborated so far is destined to be expanded: as to November 2003 it consists of 71 terms, mutually interconnected through a rich net of cross-references, and collected in the Glossary published in the first volume of this series. The terms belong to very different categories: original constructs (*algebraic babbling*, *inebriation by symbols*, *semantic persistence*); references to other theoretical constructs (*didactical contract, negotiation, pseudo-equation*); common terms used with a particular meaning (*diary, discussion, metaphor*); words belonging to the context of linguistics (*paraphrase, syntax, translating*) or to a mathematical context (*unknown, multiplicative form, equal*); adjectives that assume nuances of meaning that differ from their own (*naive, opaque, transparent*).

4. This Unit's keywords

For the reader's ease we report here all the Glossary's keywords that are referred to within the Unit; they are <u>underlined</u> the first time they appear.

Additive (form, representation)
Algebraic babbling
Arguing
Brioshi
Canonical / non canonical (representation, form)
Collective (exchange, discussion)
Describing (in mathematical language)
Diary of joint sessions activities
Didactical contract
Discussion → Collective (exchange, discussion)
Equal (sign)
Exchange → Collective (exchange, discussion)
Formal coding (writing in a formula)
Formal/formalization → translating/translation
Language (mathematics as a)
Letter (use of)
Metaphor → didactical mediator
Multiplicative (form) → Additive (form, representation)

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Notation (mathematical) → Sentence (mathematical)
Opaque / Transparent (as concerns meaning) → Procedural
Paraphrase
Process / product
Protocol
Regularity
Relationship
Represent/ solve
Representation
Result → Process / product
Semantics/ syntax
Sentence (mathematical)
Sharing → Collective (exchange, discussion)
Social (achievement, construction) → Collective (exchange, discussion)
Social mediation
Solution → Represent/solve
Spot → Didactical mediator
Syntax/ semantics → Semantics/ syntax
Translating/translation
Transparent → Opaque / Transparent (as concerns meaning)
Verbalise, verbalisation

5. The Unit

The ArAl Project Units are characterised by a constant presence of activities that entail a search for regularities, in particular in Unit 4: *Search for regularities: the numbers grid,* in Unit 5: *Pyramids of numbers* and in Unit 7: *Search for regularities: from friezes to arithmetic sequences*. Activities requiring the discovery of regularities in structures are precious for the formation of algebraic thinking, since they favour transition to *generalisation*: making pupils grasp a situation of regularity means teaching them how to identify the key for an algebraic reading of the considered *structure*.

Algebra tends to unify the study of situations that are more or less similar, beyond factors like context, type of involved elements and their numerical values: in other words algebra goes beyond those elements of diversity that hinder – or even block- a process of recognition of a common basis. Similarities are recognised through the creation of correspondences among those elements of the examined situations that satisfy the relationships linking them: this process is proper of reasoning *by analogy*.

When these correspondences are built situations are said to be *analogous* or presenting the *same structure*, or else, that they are linked by a *structural analogy*. The term structure refers to the net of relationships that connect elements involved in one particular situation. Situations are said to be analogous when they share this net.

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Searching for regularities can give a lot of information to teachers: they can understand whether pupils learn to tackle problem situations with method and systematically, whether they are able to express themselves with appropriate language (also using formulae), whether they can make predictions and verify them.

Problem situations tackled in the Unit are concrete (for instance, constructions made with matches), realistic (pupils who love organising collections of objects or challenging friends, parents, Brioshi to see whether they are able to understand the method they use).

Thus pupils must explore situations as if they were investigators who, using cues, find out how t study the situation and deduce the underlying law or, vice versa, how to decode that law to apply it consistently to analogous situations.

6. Aspetti didattici

L'Unità propone delle situazioni problematiche che – attraverso l'esplorazione individuale, di gruppo o di classe e la discussione collettiva – conducono gli alunni alla conquista della *legge* che regola la loro struttura e alla sua *rappresentazione* mediante il simbolismo matematico.

Le situazioni hanno forti supporti visivi in modo che l'aspetto percettivo possa aiutare a comprendere l'ambiente nel quale si conducono le loro esplorazioni.

Il confronto tra rappresentazioni differenti è costantemente presente perché conduce gli alunni non solo alla comprensione del significato delle scritture ma li abitua a cogliere equivalenze tra parafrasi talvolta non semplici da confrontare, soprattutto se manca l'abitudine al farlo. Questa capacità diventa fondamentale in seguito, soprattutto nella scuola superiore, quando gli studenti affronteranno situazioni che richiedono il controllo sia degli aspetti semantici che di quelli sintattici delle scritture algebriche, e dovranno saper gestire la loro manipolazione.

A questo scopo tutte le situazioni problematiche sono organizzate in modo tale da favorire il confronto fra i registri variamente intrecciati nei quali esse possono essere descritte dagli allievi all'atto della loro esplorazione: linguaggi iconico, naturale, algebrico. Allo scopo di supportare ed arricchire tali intrecci, vengono costantemente utilizzati altri strumenti di rappresentazione come le tabelle e le frecce. La lavagna gioca un ruolo importante come continuo supporto all'attività e, in particolare, alla discussione; per questa ragione nelle pagine dell'Unità viene sottolineato il suo utilizzo con una simbologia molto realistica.

7. Terminology and symbols

Phase	Sequence of situations of increasing difficulty referring
Situation	Problem around which individual, group and class ac-

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Blackboard	The frame with a black background and white signs underlines the important role played by the blackboard during collective discussions and during the exchange of messages with Brioshi.
Expansion	The grey frame contains a working hypothesis on a possible widening of the activity in an algebraic direction. Environmental conditions and teacher's objectives are fundamental factors for its implementation.
Supplementary activity	The grey frame contains an extension to topics related to those developed in previous Situations
Note	The grey frame contains either a methodological or an operative hint for the teacher
.... n	A footnote sign near a term or at the end of a sentence refers to an explanation in the right hand column of Comments.
	In the grey bordered frame a problem situation is described. The proposed text is only indicative; its formulation represents the outcome of a <u>social mediation</u> between teacher and class.
<u>Underlined term</u>	An underlined term refers back to a correspondent voice in the Glossary. The term is underlined the first time it appears in the text.
<u>Diary</u>	The frame contains a meaningful excerpt of a discussion taken from the minutes of one of the activities carried out in a class participating in the ArAl project. Some symbols synthesise the type of intervention:
√	√ Teacher's intervention
	A pupil's intervention
	Summary of some interventions
	Result of a collective discussion (a principle, a rule, a conclusion, an observation and so on).
→	Two arrows at the end of a page and at the beginning of the next page mean that the text (Diary, protocol etc.) in which they are included is not interrupted.

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8. Phases and expansions, Situations and topics

PHASES	SITUATIONS	TOPICS
First	**1 - 6**	Problems with grids made of toothpicks
Second	**7 - 12**	Search for regularities on: belts of increasing lengths; waves drawings; collections and toys boxes made of pairs of objects linked by a law invented by the owner.
Third	**13 - 15**	Exchange of challenger with Brioshi about the solution of problem situations centred on the search for regularities.

Some of the proposed situations are inspired by others that can be found in the English project *NMP mathematics for secondary School edited by* E. Harper (Longmann 1987).

9. Distribution of situations in relation to pupils' age

The distribution represents an indicative proposal based on the experiences carried out in the Project's classes.

An important aspect is whether pupils who tackle these *explorations* have already carried out activities within ArAl project, or have anyway dealt with themes linked to an early approach to algebraic thinking: this means that all depends on whether or not pupils have prerequisites referring to themes such as different representations of a number, use of letters or general aspects like a collective reflection on mathematical objects or rather approach to generalisation.

So, depending on environmental conditions, all phases – except the three Expansions (**E1**, **E2**, **E3**), suitable for junior high school pupils– can be tackled by the whole range of classes for which the Unit has been thought, from 4th to 8th grade.

It will be teacher's task, on the basis of his/her experience, objectives and features of his/her class, to evaluate how activities can be adapted to his/her own needs.

		PHASES AND SITUATIONS																	
		I								**II**							**III**		
		1	**2**	**3**	**4**	**E1**	**5**	**6**	**E2**	**7**	**8**	**9**	**E3**	**10**	**11**	**12**	**13**	**14**	**15**
pri	4																		
	5																		
second	1																		
	2																		
	3																		

The Unit

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

First phase

In this first phase pupils carry out their explorations in a very concrete environment: networks of toothpicks of different dimensions. The teacher can only use real objects (toothpicks glued on a cardboard or drawings of networks representing toothpicks by general agreement).
The first situations prepare the ground to a search for correspondence laws. They favour pupils' reflection on their own thinking processes and the capacity of making them explicit using natural language – oral and written- and mathematical language, both in autonomous and variously combined forms.
From a methodological point of view we seek clarity in the organisation of protocols, in their collective analysis and comparison, with the main objective of learning how to compare and exchange one's statements- no matters whether right or wrong- with those of companions, in order to contribute to construct a socially shared knowledge being increasingly aware.

1. Counting toothpicks

The following siuation[1] is proposed:

The drawing represents some toothpicks on top of a table. How many are there?

Pupils are requested to wait for everybody to complete their counting before intervening. Most probably the answer (19) will be very quick.

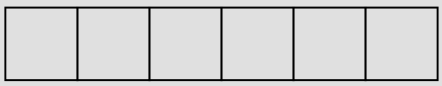
1. *All the problem situations presented within the Unit are put in grey frames (see paragraph 7: Terminology and symbolic systems) with the purpose of favouring the reader's focusing. In actual fact they are narrated by the teacher in class along with the construction of networks with real toothpicks or with drawings. They can also become worksheets, to be given to pupils after presenting the activity in class. The authors do not mean to suggest this working strategy in this place, however they deem useful that in certain situations the student can avail of an individual document on which to reflect and operate autonomously.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

2. How did you count?

The situation evolves. The performance implicit in the first task ("How many toothpicks are there?") requires the identification of a result, and therefore to calculate: it is set at a *cognitive* level.
The next task ("Explain how you did it") is more complex because it is more difficult to look at *oneself while calculating* than *calculate*: it is set at a *metacognitive* level.

Now explain as clearly as possible how you found the number of toothpicks.



Gli alunni ora devono dare la risposta per iscritto.
I protocolli vengono presentati alla classe e discussi.
L'insegnante rimarrà sorpreso dalla varietà di strategie applicate anche in una situazione così semplice (sarebbe molto utile che lui stesso verificasse la sua personale strategia).
Il prossimo diario fornisce alcuni esempi di strategie utilizzate per il conteggio e dei modi nei quali esse sono esplicitate.

Note 1: Exploration

Exploration of problem situations is important because it contributes to questioning the widespread conception of mathematics as something complete and that one cannot do anything but accept passively that someone teaches it as it is.. *Arguing around a problem situation means becoming* knowledge producers *finding out that there might be several correct strategies that although leading to the same result might be the outcome of completely different mental processes. Pupils realise that the teacher himself is involved in this activity, between creativity and organisation of thinking and that his role is not only that of transmitting knowledge, but also to co-ordinate the process of discovery, often being a protagonist himself. Pupils are thus led to understand that mathematics can be an environment in which interconnections can be established between taste for discovering, capacity of expressing and synthesising one's thoughts and exchange with others, through a positive union of individual attitudes and collective elaborations. Primary objective is to favour the form of intelligence – widely recognised as the most precious- that Howard Gardner calls* interpersonal intelligence *with an eye to the history of mathematical thought but also to any other form of human culture.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

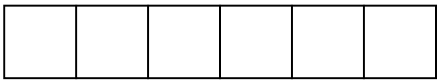
Diary 1 (5th grade, January)

Texts elaborated by pupils:

«I counted horizontal toothpicks and then vertical toothpicks and I got this operation»:

12 + 7 = 19

«I did 4 × 3 = 12 because a square has 4 toothpicks. I did 2 × 3 = 6 because there are 2 left for each square and + 1 because there is 1 left at the end»



4 + 4 + 4

12 + 6 + 1 = 19

«6 × 2 + 7. I counted horizontal toothpicks, I multiplied them by 2, since there were 2 horizontal lines, and I added 7»

«Toothpicks are 19, I did a shape like that of an 8 on a computer **2**, not to create problems I left 2 dashes, then with the other three I summed them up and I got 19»

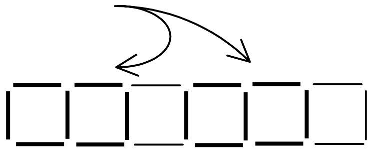
«I used 19 toothpicks:

3 × 6 = 18 18 + 1 = 19,

I counted as a 'C' **3**»

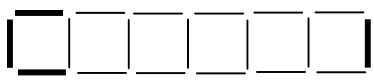
«I realised that with two squares there were 7 toothpicks, then I added 1 and I got 8. Then I multiplied by 2 and I added 3 and I got to 19». **4**

2. *The pupil perceived two '8's in their digital representation,*

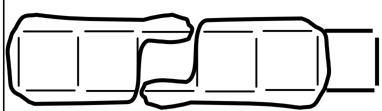


separated by the two central dashes to which he added the three dashes at the right.
Fancy readings like this one are often induced by a *<u>didactical contract</u>* *inviting pupils to free expression: pupils thus think they must 'perform' in ways that are generally not valid.*

3. *The pupil made a more productive reading because he unconsciously identified a* *<u>structure</u>* *of the drawing, premise to an algebraic representation. This reading will be analysed in depth with the class at a later moment.*



4. *The pupil shows he has 'seen' the drawing as made of two equal figures, each of which consisting of two squares and a horizontal line (we might say that one can be transformed into the other through a central symmetry). Finally he added the final three toothpicks at the right.*

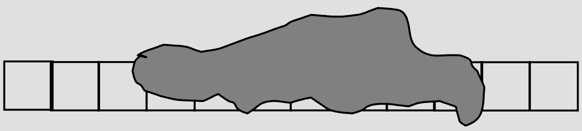


Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

3. Towards generalisation

An analogous situation is proposed; this time a spot enables counting only part of the toothpicks.
This is the problem:

Caterina constructed another rectangle with toothpicks, but Luisa dropped some ice cream on it.

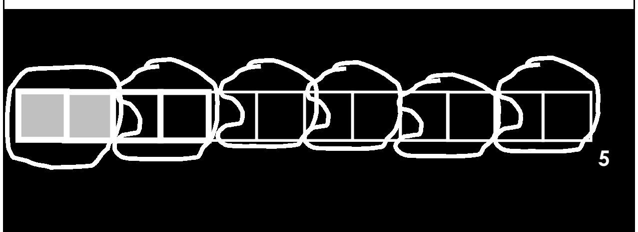


Can you find how many toothpicks she used?
Explain your own solution.

The number of toothpicks can be found only starting from the dimension of the rectangle basis, which is 12 toothpicks long, and this is the only accessible datum.

Diary 2 (5th grade, January)

«I counted the first 'C', and then I multiplied by the number of Cs. Since I cannot see under the spot I tried to imagine. Cs are 12. 3 × 12 = 36. Then one is left, and so 36 + 1 = 37»
«The first two cells have 7 toothpicks, but going ahead the vertical line closes up the other cells and you must not count them. So we get 7 + 6 + 6 + 6 + 6 + 6 = 37»
A drawing on the blackboard illustrates this reasoning.

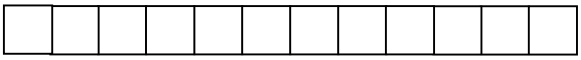


→

5. The pupil does not draw on the multiplicative model; in counting he follows a procedural way by pairs of 'C' (as he indicated on the blackboard) counting 5 of them.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
The first two cells are coloured and the initial six toothpicks are highlighted.
(he did the drawing again without the spot)



«I used 37 toothpicks. I counted them as a C to be sure they are 37, then I did 12 + 12 which were the bottom toothpicks, that is the horizontal ones, which is 24, and then I counted the vertical ones which were 13 and 24 + 13 = 37» **6**
«I tried to count vertical toothpicks and I tried to imagine how many there were where the spot is: they are 13. After that I counted horizontal ones which are 12. Then I multiplied by the two sides and I get 24 and finally I summed up 13 + 24 = 37»
«I managed to count **5** bottom toothpicks and they were 13, so also above ones had 13. I knew that the middle ones had 1 more and I did 13 × 2 + 14 = 40» **7**

4. Achieving generalisation

The class is now ready to express a relationship between the number of toothpicks of the rectangle's basis (representing its length) and the total number of toothpicks, which allows them to calculate this latter number as a function of the former one.
Next problem can be formulated as follows:

When we constructed a rectangle with 6 toothpicks in the basis, we used 19 toothpicks altogether.
When we constructed a rectangle with 12 toothpicks in the basis, we used 37 toothpicks altogether.
Can you find a rule working for any number of toothpicks in the basis?
What if we wanted to construct a Record rectangle having 1000 toothpicks in its basis? How many toothpicks would we need altogether? **8**

6. *Protocols of this type are frequent: pupils calculate in a certain way and then they provide explanations relating to a process that differs to the one they actually followed. This might depend on the difficulty linked to carrying out a metacognitive task, requiring an aware control of calculations. For example it might happen that the pupil applies two counting modalities, to have control of what he has done initially, and then makes explicit one of the two. It might also happen that the pupil spontaneously activates an elementary strategy; then when it comes to argumentation he tries to identify more 'mathematical' strategies, more rewarding with respect to the task.*

7. *Regardless of the mistake in counting the toothpicks in the basis, it is interesting that the pupil gave the result through an indication of the followed process (12 x 2), differently to the related canonical form (24), which makes it opaque.*

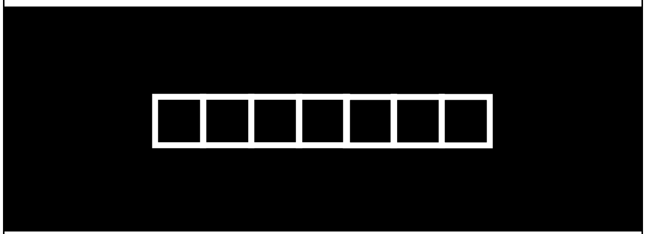
8. *Answer to the last question requires the identification of the general law and a process of making it particular in the case of a 1000 toothpicks long basis .*
In order to help pupils identify the law, a comparison between two or three cases would be useful, so hat pupils can grasp what is kept and what varies: this facilitates them to frame the situation in general terms.
In our case, comparing a rectangle with basis 7 with rectangles with bases 8 and 9 and applying the same calculation procedure – for instance that view-

Activities suitable for the classes	*1*	2	3	**4**	**5**	***1***	**2**	**3**	**Comments**

The following is an attempt to generalise:

Diary 3 (4th grade, March)

√ The following drawing is made on the blackboard



and the following task is proposed:
«Write down in mathematical language for <u>Brioshi</u> the method to find how many toothpicks you need for a rectangle which is any number of squares long ».
After a phase of incertitude the situation is unravelled; it is very rich but complex at the same time.
We transcribe proposals on the blackboard:

(a) With 31 toothpicks I make 10 squares
(b) 3 × 7 + 1   2 × 7 + 1   3 × 14 + 1
(c) 3 × n + 4
(d) 7 × 2 + 8 = 22
(e) 3 × n + 1
(f) 11 × 2 + 1 × 10 + 1

√ To start with, we ask the class what they think about the three <u>writings</u> (b).
Some pupils are extremely convinced that they are completely different.
√ We discuss about the meaning of numbers 7 and 14 in (b).
They represent the number of squares you can get and the total number of toothpicks.
√ «And what do you think about (e)?»

→

ing the figure through as many 'C's as the toothpicks in the basis plus 1 closing the figure – we realise that the total number of toothpicks is given by three times C plus 1.
The <u>translation</u> in algebraic language is immediately done, once the symbols expressing these numbers are introduced.
Indicating with 'n' the number of toothpicks in the basis, the total number of toothpicks is therefore

$$3 \times n + 1$$

or the equivalent

$$n \times 3 + 1$$

or else,,

$$3n + 1$$

Thus the rectangle '1000 × 1' needs 3001 toothpicks.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
Some recognise that n means 'any number'.
√ We ask pupils which <u>sentences</u> might be erased because they 'confuse' and Brioshi would not accept them.
They erase (a) [9] because it is written in Italian language; the second of (b)s because a higher number of toothpicks are used; (c) – although a correct basic idea is recognised - ; (f) because it does not refer to a correct counting <u>procedure</u> and actually it does not give the exact number of toothpicks.
There are still on the blackboard:

(b) 3 × 7 + 1 **3 × 14 + 1**

(d) 7 × 2 + 8 = 22
(e) 3 × n + 1 **see Expansion 1**

Some suggest they could eliminate also the second of (b)s because it does not give the correct number of toothpicks; (d) is eliminated because pupils focused on (e).
The indicated formulae are erased and only the following are kept:

(b) 3 × 7 + 1

(e) 3 × n + 1

The class is struck by the analogy between the two writings.
√ The teacher asks what n means in the 'mysterious formula' (e).
«The number of squares»
«The number of toothpicks in the basis»
→

9. *The answer is not appropriate, but the sentence (a) actually reverses the initial problem and turns it into the inverse one: 'How many squares can you make with a given number of toothpicks'? The class will go back to this theme soon.*

Expansion 1: Comparing representations

Comparing writings suggests ideas for activities that may favour the identification of equivalences between representations that are <u>formally</u> different and understanding of the power of <u>syntactic</u> transformations.
A suitable example for junior high school might be formulated as a message by Brioshi , whose class would be working at the same problem as in Diary 3 (the two writings are taken from those on the blackboard)
Suppose that Brioshi sends the following message:

3 × 7 + 1
7 × 2 + 8
?
3 × 7 + 1 = 7 × 2 + 8

Brioshi asks them to verify an equality.
The reader is invited to think about the situation before carrying on reading .
The problem may be tackled, through group activities and subsequent collective <u>discussion</u>, drawing on knowledge about canonical and non canonical forms of numbers, so that comparable writings can be obtained.
The following might be the most productive approach:
→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
√ Pupils are asked what that 'magic formula' enables them to find **10**
«The total number of toothpicks!»
√ The teacher proposes that they represent this conclusion for Brioshi.
The following writing is achieved:

$$3 \times n + 1 = s$$

√ Pupils are asked to verbalise the formula.
Linguistically different formulations are obtained:
«The number of toothpicks is 3 times the number of squares plus 1»
«I find the number of toothpicks multiplying 3 by the number of squares I want to construct and adding 1»
We decide to send the last formula to Brioshi.
A new problem is presented
√ «Very good! And now a difficult task: knowing the number of toothpicks, can you find out how many squares can be constructed? Can you explain it to Brioshi?».
Pupils are very uncertain.
Three girls propose three writings which are transcribed on the blackboard:

(g) 22 : 3 - 1
(h) n : 3 + 1
(e) 22 : 4 + 7

They discuss the three proposals. **11**
During the author's explanation (e) turns out to be not consistent and is erased.
(h) is difficult to be interpreted. (g) is similar but more 'concrete' and favours interpretation.
Some pupils get to the intuition that 1 must be taken out before dividing (they get 'C's again).
They get to formulate the following writings:

$$(22 – 1) : 3$$

The achievement of this conclusion is satisfactory.

→
7 × 2 + 8 =
you rewrite 8 in non canonical form
= 7 × 2 + (7 + 1) =
you take out parentheses
= 7 × 2 + 7 + 1 =
you rewrite 7 in non canonical form:
= 7 × 2 + 7 × 1 + 1 =
This enables the application of the distributive law:
= 7 × (2 + 1) + 1 =
that can be rewritten:
= 7 × 3 + 1
Concluding:

7 × 2 + 8 = 7 × 3 + 1

All this can be summed up in the answer for Brioshi:

7 × 2 + 8 =
= 7 × 2 + (7 + 1) =
= 7 × 2 + 7 + 1 =
= 7 × 2 + 7 × 1 + 1 =
= 7 × (2 + 1) + 1 =
= 7 × 3 + 1
↓
7 × 2 + 8 = 3 × 7 + 1

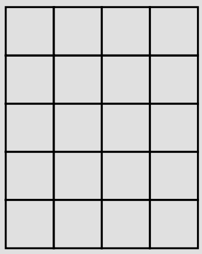
10 *Interestingly, due to the simple figure, generalisation is obtained operating only on rectangle 7 × 1. In more complex figures this does not happen and different cases are to be examined in order to grasp analogies.*

11*. The problem is not simple because pupils must realise that, in inverting a formula, inverse operators swap. If you first multiply by 3 and then add 1, when you invert, you first add 1 and then divide by 3. Moreover, the latter operation requires the introduction of parentheses.*

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

5. A grid of toothpicks

Gaia constructed this figure with some toothpicks:



Explain the way in which you find how many toothpicks she used. 12

12. The reader is advised to solve personally the problem before continuing; the reading of diary protocols will be much clearer

Diary 4 (5th grade, March)

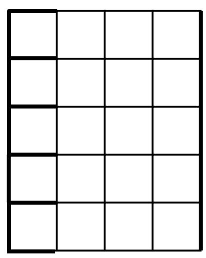
Pupils dictate their proposals:

(a) 4 × 6 + 5 × 5 (3 pupils)
(b) 11 × 4 + 5
(c) 2 × 20 + 5 (4 pupils)
(d) 4 × 20 : 2 + 10 – 1
(e) 25 + 24 (3 pupils)
(f) (4 × 4) × 4 + 5
(g) 4 × 20
(h) 16 × 3 + 1
(i) 9 × 5 = 45 + 4 = 49
(j) 24 × 2 + 1
(k) 16 × 16 + 5

It is decided to check for the correctness of results and to erase wrong expressions.13

13. The following drawings reproduce those made by pupils during the activity.

(b) 11 × 4 + 5



(a) 4 × 6 + 5 × 5 is correct; it represents the followed strategy very clearly;

(b, see drawing) ‘11’ represents the ‘comb-shaped stamp’ which is repeated 4 times and that the 5 are those left (see drawing)

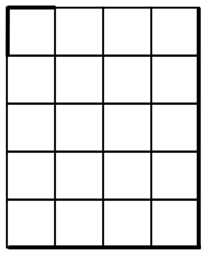
→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

→
(c, see drawing) is wrong; but when the author explains it is clear that the reading of the drawing is correct (twenty upside-down 'L's plus the right hand column); a detail is missing though and it is added by some pupils (the 5 toothpicks of the first bottom row) (see drawing); the pupil himself modifies his writing into:

$$2 \times 20 + 5 + 4.$$

(c) 2 × 20 + 5



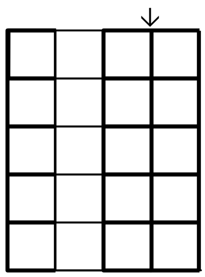
(d) 4 × 20 : 2 + 10 – 1 is hard for pupils also because the author does not help them to understand; it is nevertheless increasingly interesting as the situation gets more defined.
It does not represent a 'formula' (as 'translation in mathematical <u>language</u> of a method for counting toothpicks) but a *story* from which the hard progressing of the author's calculation emerges, including his regrets.
We explain it through a synthesis of the pupil's words in his oral explanation.
First he saw each square as formed by 4 toothpicks; then he counted the 20 squares and multiplied 4 x 20. At this stage he realised that some toothpicks were counted twice (perhaps the 'inner' ones, but we are not quite sure) so he divided 80 by 2 and got 40. In this way the toothpicks were few and he added 10 (we understood: those corresponding to half border), until he realised that they are not 10 but 9 and he took 1 away.

(e) 25 + 24 is like (a) but more opaque.

(f) (4 × 4) × 4 + 5 and (g) 4 × 20 are wrong and anyway the authors could not explain them.

The author of h (see drawing), in attempting to explain his strategy shows a strongly disorganised 'reading' of the drawing. He counted toothpicks in the first column (16) and multiplied 16 by 3. He realised that in this way the 5 vertical toothpicks indicated by the arrow were counted twice and 6 horizontal toothpicks were missing. He thought that he had already counted 5 so he added only 1.
→

(h) 16 × 3 + 1

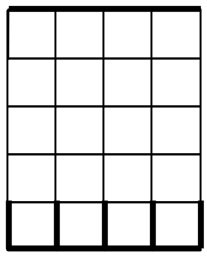


Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

→
(i, see drawing) The class observes straight away that the representation contains the classic 'chain of equals' error; the intention is anyway clear. It reflects a pattern perception analogous to that of (b).

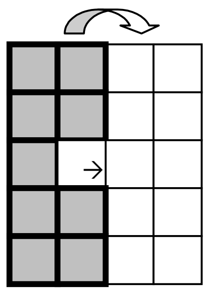
In (j, see drawing) pupils saw a central symmetry (the part that rotates is highlighted in grey); then the central toothpick was added (in the drawing it is indicated by the arrow).

(i) 9 × 5 = 45 + 4 = 49



In another 5th grade the same problem was solved individually in writing. Some of the answers classified by typology are reported here.

(j) 24 × 2 + 1



Diary 5 (5th grade, March)

Classification of protocols:

a) Strategy 'by rows and columns'

- I counted how many toothpicks are in vertical position and I calculated 5 × 5, then I counted toothpicks in horizontal position and I multiplied them by 6, after that I added up.
  5 × 5 = 25   4 × 6 = 24   25 + 24 = 49
- I counted toothpicks lying in a column and I did × 4 because there are 4 vertical lines, then I counted those in vertical and I did × 5 because there are 5 horizontal lines, then I added up the two results and I got 49.
  To. lying vertic.lines   To.vertic. horiz. lines
  (4 × 6)   +   (5 × 5) = 49

b) Strategy 'border and inner'

In order to find the total amount first I counted the perimeter, making (5 × 2) + (4 × 2) which is 18. Then I counted the number of columns and I multiplied it by the number of toothpicks (3 columns, 5 toothpicks) 3 × 5 = 15 which, added to the number of lines times the number of toothpicks in each of them (4 rows, 4 toothpicks, 4 × 4 = 16) gives a total of 31 toothpicks (15 + 16) which, added to the perimeter, gives 49 (31 + 18).

→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
c) Strategy '1 + 1 + 1 + 1 +…'
- I counted one by one and I got 51, but then I counted them again and they were 49. I counted them again and they were 49. Therefore the correct result is 49. I counted one by one and I got 51,

d) Mistakes
- I used 49 toothpicks and I found them counting them on the grid, but I made several trials which were wrong using the following data: 4 × 5 perimeter (18) area 20 squares and 4 sides in each square.
- I tried to do this: 20 × 4 = that is 20 squares and 4 toothpicks each square 80 toothpicks. **14**

The same situation was proposed in lower secondary school classes, some of which are involved in a virtual exchange with foreign almost-peers.
The English researcher Dave Hewitt had proposed the same problem with toothpicks to 15-16 years old pupils and published results in an article which reported a great number of commented protocols. **15** Lower secondary school classes of ArAl Project engaging in the same situation carried out their work in two phases: first they elaborated some calculus hypotheses and then they interpreted the mathematical writings produced by English students, all in a classic 'Brioshi environment'. **16**
Among lower secondary school protocols, substantially similar to primary school ones, we report one which differs from all others for the attempt to generalise:

> To find the number of toothpicks I counted first toothpicks in the 'shorter side', I multiplied them by those in the 'longer side' twice and then I added half perimeter.
>
> $a \times b + a \times b + a + b$

14. *The pupil does not realise he counted a great number of toothpicks twice. Differently to protocol (c), no control is enacted.*

15. *Hewitt, D. (1998), Approaching Arithmetics algebraically, MT, 163: 19-29.*

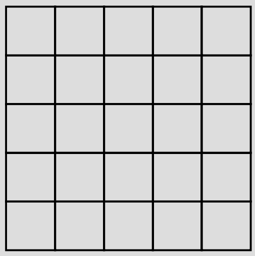
16. *Strategies elaborated by Italian students, both from primary and lower secondary, and by English secondary school students show an extraordinary variety of points of view, not depending on age or nationality, and many of them are really similar. We believe that activities requiring creative exploration are relatively more frequent in early school years and decrease as students' age increases. In later stages other, more technical activities are privileged and therefore important competencies are not valued and performances end up being very similar.*

Activities suitable for the classes	*1*	*2*	*3*	***4***	***5***	***1***	***2***	***3***	**Comments**

6. To interpret strategies expressed in mathematical language

This situation is proposed:

Alberto, Barbara and Carlotta have counted the toothpicks necessary to make this drawing:



They described their strategies as follows:
a) 5 + 5 × 11
b) 3 × (3 × 5 + 1) + 6 + 6
c) 4 × 5 + 2 × 4 × 5
Interpret strategies trying to understand how each of them 'saw' the drawing made of toothpicks.

Diary 6 (4th grade, April)

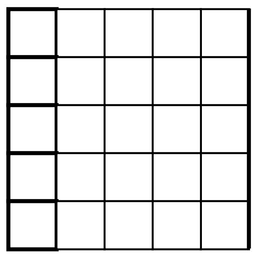
Discussion leads to these conclusions:

a (see drawing) Each column is made of 11 toothpicks, so you calculate 5 × 11. Then you add 5 toothpicks vertically.

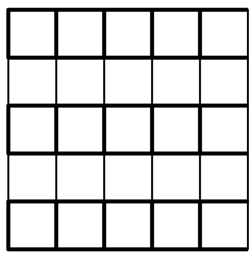
b (see drawing) Each row is seen as if it were made of 5 'C's (3 × 5) and of a final vertical toothpick; in this way you calculate toothpicks in the first, third and fifth rows (this explains the initial '3 ×'). Then you add the 6 vertical toothpicks of the second row (+ 6) and those of the fourth row (+ 6).

→

(a) The lines in bold indicate the 11 toothpicks in each column and the last 5 to the right.



(b)



Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
c (see drawing) 4 × 5 represents the number of toothpicks in the border; 4 rows are still to be calculated, each made of 5 horizontal toothpicks and 4 columns, each made of 5 vertical toothpicks (2 × 4 × 5). **17**

Another writing of reasoning (c) is proposed which is equivalent but more 'consistent':

5 × 4 + 5 × 4 × 2

We propose now an **Expansion** that allows a widening of the exploration of the toothpicks square presented in Situation 6 towards a generalisation and the conquest of a *rule*.
This develops through a sequence of stages:

(A) individual analysis of the situation;

(B) individual search for a method for counting toothpicks;

(C) representation of the reasoning through mathematical language;

(D) collective illustration of methods.

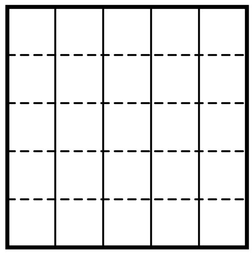
The next level, suitable for older pupils (7th -8th grades), is illustrated in two Expansions, in which aspects of the first level are extended, thus leading towards:

(E) generalisation;

(F) algebraic representation;

(G) formal manipulation.

(c)



17. *The writing 4 × 5 does not convince part of the pupils who would have considered 5 × 4 as more correct. Of course from a mathematical point of view the two writings are equivalent but the second one translates the situation more faithfully: in order to get toothpicks it is more logical to think that you multiply the number of toothpicks (multiplicand) by the number of rows (multiplier). We deem extremely appropriate for pupils to reflect on all these nuances in meaning. A deeper discussion of these aspects are in Unit 1, Brioshi and the approach to algebraic code.*

Activities suitable for the classes	1	2	3	4	5	1	**2**	**3**	**Comments**

Expansion 2: towards generalisation

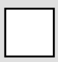
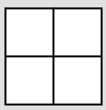
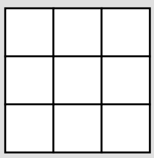
After Situation 6 this continuation is proposed:

> Find a rule enabling you to recognise the number of toothpicks needed to construct a squared grid made of any number of squares.

The class can be left to search for a strategy and possibly directed towards a group exploration of the way in which the number of toothpicks changes in square grids, made of squares, with increasing sides starting from one toothpick.
Pupils are supposed to be able to insert data in a table and know when it is useful. Also the choice of variables to be included in the table is made together with pupils. These will be indicated with a letter:

q number of squares
s number of toothpicks of the side of the square
f total number of toothpicks

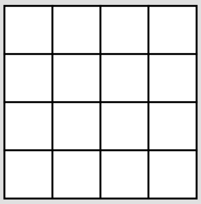
When all groups will have finished counting their respective squares data will be recorded:

	q	s	f
	1	1	4
	4	2	12
	9	3	24

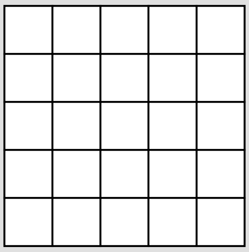
→

Activities suitable for the classes	1	2	3	4	5	1	**2**	**3**	**Comments**

→



16 4 40



25 5 60

and so on.
At this point it is fundamental that pupils know that a number can be represented in various different ways, and that non canonical representations can be helpful in identifying a law.
The class must be led to draw on this knowledge and apply it. A table similar to the following is elaborated at the blackboard:

q	s	f
1	1	4
4	2	12
9	3	24
16	4	40
25	5	60

Pupils are invited to express ‘f’ as a function of ‘l’ or of ‘q’; the first relationship is probably identified more easily:

q	s	f	
1	1	4	1×4
4	2	12	2×6
9	3	24	3×8
16	4	40	4×10
25	5	60	5×12

→

Activities suitable for the classes	1	2	3	4	5	1	**2**	**3**	Comments

→
The step made is important but not sufficient; also 4,6,8,10 and 12 must be expressed as functions of ‘s’.
Another representation is obtained, for instance this one, that leads to a fundamental expression for the table:

q	s	f		.
1	1	4	1×4	1×(1×2+2)
4	2	12	2×6	2×(2×2+2)
9	3	24	3×8	3×(3×2+2)
16	4	40	4×10	4×(4×2+2)
25	5	60	5×12	5×(5×2+2)
q	s	l×12		s×(s×2+2)

The general law is thus obtained:

(a) $f = s \times (s \times 2 + 2)$

This can be particularised, for example, by posing a question like:

How many toothpicks are needed to construct a square with a 30 toothpicks-long side?
Pupils understand that in order to get this number they must substitute number 30 for s in the formula. This means

$$f = 30 \times (30 \times 2 + 2) = 30 \times 62 = 1860$$

In this way they establish that 1860 toothpicks are needed.
It is important to translate the law into natural language through a collective comparison of definitions; one possible formulation might be:

‘The number of toothpicks to construct a square equals the number of toothpicks of its side multiplied by twice the side plus 2.’ **18**

It might happen that some pupil (or group) identify other relationships, such as:

→

***18.** The translation is linguistically ‘heavy’ (although it is quite synthetic); this may favour a reflection on the meaning of formal language: it is more synthetic and <u>economic</u> than natural one.*

Activities suitable for the classes	1	2	3	4	5	1	**2**	**3**	**Comments**

→

q	s	f		
1	1	4	2×2	2×1×(1+2)
4	2	12	4×3	2×2×(2+1)
9	3	24	6×4	2×3×(3+1)
16	4	40	8×5	2×4×(4+1)
25	5	60	10×6	2×5×(5+1)
q	s	l×12		2×s×(s+1)

They would probably claim very proudly that they have found *another law*. **19**

Pupils are invited to compare the two writings: do they say the same thing?

$$s \times (s \times 2 + 2) = 2 \times s \times (s + 1)$$

Some representations are modified…

(b$_1$) $s \times (2s + 2) = 2 \times s \times (s + 1)$

(b$_2$) $2s^2 + 2s = 2s^2 + 2s$

… and they find out that the law is always the same! And that it can be written in other different ways; for example, by applying the distributive law to (b$_2$):

(c) $2s^2 + 2s = 2s \times (s + 1)$

Or rather, by applying the distributive law to the first member of (c):

(d) $2s^2 + 2s = 2 \times (s^2 + s)$

The situation is very rich because it allows dealing with a number of themes: the meaning of algebraic writing and related manipulations, recognition of equivalent writings and hence the capacity to cope with paraphrases , the use of laws, such as the distributive law, also in its version of ‘common factors extraction’ as in transitions from (b) to (c) or to (d).

19. *Often searching for regularities pupils, being inexperienced, get to identify ‘local truths’ and they exhibit them as laws, i.e. as ‘global truths’, without verifying them. It is important that they understand that this route can be dangerous.*

For instance, in a 8th grade class a group found this elegant representation for values s = 5 and f = 60:

$$f = 3s\,(s - 1)$$

generalisation of:

$$60 = 3 \times 5 \times (5 - 1)$$

The teacher himself was struck and in a first moment had accepted the writing. Some pupils, though, (fortunately) maybe for a sort of jealousy,, verified that the formula was not true for other cases. This allowed for a discussion on the local-global dichotomy and gave a chance for more numerous verifications.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

Second phase

The activity proposed at this stage, like the previous one, develops at different levels, and the teacher can decide freely whether and when to interrupt it, continue it or think about changes.
The structure of problems – search for correspondences in collections of objects – is constant: there are two sets of elements, mutually connected by a law and the objective is to find this law, trying then to deduce the inverse from it.

Note 2: Instruments, strategies, objectives.

We deem appropriate to synthesise *instruments*, *strategies* and *objectives* in order to set up a sort of ship's equipment for navigating through the difficulties of next activities.

The main instruments

- *The table*: it is a very powerful instrument for organising data; through their analysis pupils get used to exploring the possible relationships linking them. It obliges pupils to use *non canonical* representations in an 'intelligent' way, in order to transform numbers with an *opaque* meaning into *transparent processes*. It gives pupils the chance to neatly highlight what changes and what stays constant in relationships between different cases thus favouring a transition to generalisation and to identification of the *correspondence law* in one or more formulations.
- *Arrow representation:* arrows initially favour the intuition of dynamic aspects of processes, highlighting their time features and therefore favouring concentration on the progression of phases of a calculation ('first I multiply, then I add, then I divide' and so on).

→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→

In a second moment they help to identify the inverse process, which is something abstract, to be 'achieved', differently to the direct process. In this sense, arrows act as <u>mediators</u> between the space-time character of concrete experiences and a distant attitude toward the object, in order to evaluate it being detached from what was done during the 'doing time'. Therefore arrows support the maturing of metacognitive attitudes.

- *The blackboard:* it works as a support to the discussion, almost becoming an interactive instrument. On the blackboard pupils write the most representative sentences elaborated by the class during the solution of problem situations, using different registers –natural, algebraic, iconic language etc. ; different proposals are amended, modified on cancelled during collective exchange; the most significant relationships are highlighted and analysed; tables are constructed; hypotheses are formulated; there is an exchange of messages with Brioshi and so on.
- *The Yes and No Game:* it is a very powerful didactical tool and it is best using it when the objective is to make students identify a solution or justify it on their own. Let us synthesise its structure: we propose a problem situation requiring as solution the result of a calculation that is consistent with a certain process. Attempts made by pupils, depending on their correctness are reported either on the 'yes' or on the 'no' column by the teacher (or by a pupil). 'Who knows' reports proposals without commenting on them. Those who dictated correct solutions must keep silent. The two extending lists are the only cue for the achievement of the correct solution and its justification.

Strategies

- *Systematic analysis*: it is the key element for developing the activity; its aim is the identification of relationships between elements of a certain problem situation.

→

Activities suitable for the classes	*1*	2	3	***4***	***5***	***1***	***2***	***3***	**Comments**

→

- *Discussion*: represents – so to speak – a fertile round for the activity; it allows each pupil to make the class aware of his/her ideas, to exchange ideas and select the most meaningful ones, to correct mistakes. It is the key instrument to construct *socially shared knowledge*.
- *Verbalisation and argumentation*: they allow pupils to describe hypotheses, calculations, processes. They represent mediation between *inner language* and the achievement of *symbolic representation*.
- *Translation*: it is the delicate and powerful moment of the transition between two different linguistic codes; it stimulates attention on the ways in which language is constructed and on analogies and differences between languages often characterised by a lack of comparability, at a surface reading. This favours the achievement of the sense of a writing in its different representations and of its semantic and syntactic aspects.

Objective

- *Search for relationships*: identifying relationships between elements of situations that are linked by a common *law* and representing them in the various languages – natural, arithmetic, algebraic, iconic, etc. means learning to move from being incapable of interpreting *details* with no meaning, to a *global* view that can be translated in mathematical language. This also enables pupils to discover invariant elements characterising an entire class of similar problem situations.

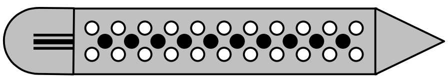
To illustrate the content of Note 2 three problem situations are proposed, centred on the themes:
- belts
- waves
- collections of objects.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

These three situations are on the same level: there is no real progression from one to the other and in fact they belong to the same phase. Each of them – with its Diaries – exemplifies the methodological situation summarised in Note 2 and highlights the use of instruments – tables, arrows, blackboard- and strategies centred on verbalisation, argumentation and translations between different linguistic codes.

7. Arabella's belt: white spots and black spots.

Arabella's belt is presented; it is made of coloured cardboard and it is about 25 cm long. It is shown to the class which observes it carefully.



The first request is to <u>describe it</u> and a discussion is enacted.
Often – as Diaries 5 and 6 show – the description is initially purely qualitative and pupils dwell on shape and colours. A patient work of directing their attention toward relationships between numbers of spots on the belt is needed: moreover pupils are expected to mix the two aspects in their remarks. Slowly, meaningful relationships in mathematical terms are put forward and direct the development of the collective reasoning.

Diary 7 (5th grade, November)

«It is little, short with several holes »
«It has three lines of holes, the central line is black and the side ones white»
«It is spiky in the front and rounded behind»
«There are two parallel lines »
«There are three»
✓ «What information do you consider more important?»
«It has three parallel lines of holes, a black one in the middle and 2 white ones»

Note 3: A priori analysis

The reading of Diaries is far more efficient if it is preceded by an a priori analysis of the proposed problem situations and related tasks.
It is important that this analysis is carried out in terms of both difficulties possibly met by the reader and those that pupils might encounter if the activity was proposed in the classroom.
Moreover, an a priori analysis is suitable because the complexity of problem situations often leads the class to rich but not linear collective discussions. This requires an effort to interpret interventions reported in the Diaries. A priori analysis, therefore, is helpful to capture consistencies and inconsistencies in discussions and the higher or lower correctness of definitions and writings in algebraic language produced by pupils.
The chosen Diaries are included in the Unit merely to favour reflection on class dynamic processes and on ways in which pupils are led toward the achievement of a certain concept, within that system of discoveries, intuitions, mistakes, exchanges of ideas that we called algebraic <u>stuttering.</u> A priori analysis of the situation is the basis for an a posteriori analysis, which is equally important, since it leads to evaluate one's own initial and ongoing reflections, as well as the progressing of the activity as it is described in the Diary.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

Diary 8 (4th and 5th grades together, November)

- «It has coloured and non coloured spots »
- «They are always laid out in the same way »
- «It has a beginning and an end »
- «The black ones are half (the white ones)»
- «There are three lines of dots »
- «The two white ones with the black one form a triangle »
- «They look like the five dots of dice »

To favour reflection on the object it is convenient to ask pupils to draw it. Observing pupils while they are engaging with this task we realise that he technique used to construct a representation – the use of instruments, initial point of the drawing, the way in which the drawing is located on the sheet, the relationship with the squares and so on – impacts the pupil's interpretation of the representation itself to an extent that is not predicted by the teacher.
About this it is useful to insert here a diary in which trainee students of a school for teachers participating in the ArAl project observed, classified and commented upon strategies they could observe when pupils were drawing.

Diary 9 (5th grade, October)

- Only two children start drawing the black spots, then they draw the white spots and finally the border.
- The bulk of children start drawing the border and then they find it hard to make spots fit in it.
- The belt is drawn horizontally, vertically or diagonally: the three directions are equally represented in the class.

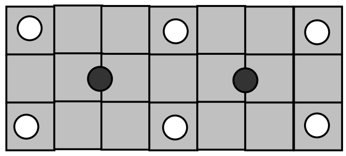
→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

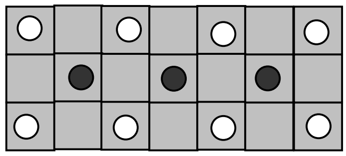
→
- Children who chose two squares as the distance between two spots are in greater difficulty because the next line of spots must be drawn in between the rows and this causes confusion. [20]
- Another difficulty is represented, in belts drawn diagonally, to follow the direction correctly.
- Children are very precise, almost all of them use a ruler and pay much attention to details.
- Some pupils, mainly those who drew the border first, end up with the black spot.

After class observations have been guided toward more productive points of view, a 'mathematical exploration'. Very probably in the beginning pupils will count the spots in Arabella's belt (they are few and it is rather quick doing it) and make mental calculations, or 'disguise' analyses based on qualities of the belt's elements as quantitative analyses. Often hurry in calculations leads to mistakes and this slows down or diverts the class investigation.
Pupils typically focus on the total number of spots, i.e. on the final results of a calculation. The productive strategy- identifying *relationships* between the number of white spots and that of black spots – is far more complex on the conceptual plane. These types of problem situations are precious because they educate to <u>relational</u> thinking.

***20.** The drawing illustrates the obstacle:*



Pupils who choose the distance of a square obtain 'clean' drawings:



Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

Diary 10 (5th grade, October)

«It has three parallel lines of spots, a black one in the middle and 2 white ones»
«There are 30 spots, 10 black and 20 white »
«They are 32 (counting spots aloud)»
«I had counted the black line and then I multiplied by 2»
«White lines have 11 spots whereas the black one has 10»

Diary 11 (5th grade, November)

√ «How many spots are there in Arabella's belt?»
«25!»
«32!»
√ «Oh, for Heaven's sake! Shall we focus?» she calls a pupil and invites him to count spots aloud.
«I count the white ones times two and the black ones one by one» [21]
√ «How is it convenient to count these spots?»
«I count by 5 and then I count the black spots which were left out»
«The calculation he said, 5 by5, does not work because there are 32 spots»
√ «Write down the calculation procedure you would send to Brioshi»
Pupils work individually. They write these sentences on the blackboard:

(a) 5 × 5 + 5
(b) 11 × 2 + 10
(c) 2 × 11 = 22 + 10 = 32 [22]

√ «What do you think about (c)? Do you think it is correct?»
Discussion helps pupils to remind concepts dealt with in previous years and the sentence is modified in the following way:
→

21. *He means that he multiplies by 2 the number of white spots in a line. In these cases it is better to intervene and invite pupils to express themselves with a clearer language.*

22. *This is a typical writing and derives from a weak control of the meaning of equal: the concept of 'equivalence' is made opaque by the operative one, which is much stronger ('equal' is considered as a '<u>directional operator</u>'). We believe that some situations favour the former point of view:*
- *using numbers written in non canonical form, e.g.:* $6 – 2 = 12 : 3$
- *using writings formed by several numbers, e.g.:* $15 – 7 + 2 = 13 + 3 – 6$
- *writing the number in canonical form to the left of the equal sign,, e.g. .:* $9 = 15 – 2 – 4$
- *setting up 'chains' of equalities, e.g.:* $3 + 9 = 6 × 2 = 10 + 5 – 3$
- *using the symbol '≠', e.g.:* $7 + 5 ≠ 3 + 10$

Clearly these aspects are to be dealt with as soon as possible, since 1st grade.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→

(c) 2 × 11 + 10 = 32

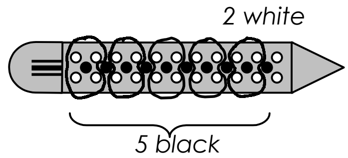
(a) is considered wrong.
Pupils keep searching for messages for Brioshi.
«5 × 5 + 5 + 2»
√ «Can you let us understand where you see 5 in the belt?»
The pupil points to the group of 5 (4 white and one black), the five black spots not included and the last two white spots. **23**
«She could also say 5 × 6 + 2» **24**
«5 × 5 + 7»
«9 × 3 + 5» **25**
√ «Which messages are clearer in your opinion?»
«11 × 2 + 10 and 2 × 11 + 10»
√ «what's the difference between them?»
«no one»
√ «It's true, but I think that the authors of these two messages counted spots in different ways. Do you reckon how they did?»
«He wrote 2 × 11 + 10 because he saw 2 white spots on top of each other and he counted them 11 times and then he saw the 10 black spots. Over there, where it's written 11 × 2 + 10 she counted 11 white spots, saw the other equal line, multiplied by 2 and added the 10 black spots»

Diary 12 (4^{th} and 5^{th} grades together, December)

√ the teacher asks what information is necessary to count spots in the belt.
«We need name of the owner, number of white spots and number of white spots »

→

23. *the pupil perceives the situation in this way:*



24. *It is interesting to compare the writing*

5 × 6 + 2

with the one she means to substitute

5 × 5 + 5 + 2

The latter one synthesises a story: the author 'sees' spots in a certain way and organises them in his mental space. Then he tells a story of how he puts them together placing them in a space-time frame: 'I multiply 5 by 5, then I add 5 and finally I add 2'.
The former one locates at an abstract level, since it represents a formal elaboration that does not take into account the perception of the object. Space and time do no longer interfere, the writing 5×5+5 was compacted into a synthetic 5×6, and this is the result of a metacognitive reading.
This confirms the hypothesis that reflecting on one's own perception and its representation, as naive as it can be, in mathematical language represent a meaningful initial moment – delicate and powerful at the same time- toward abstraction.

25. *Now pupils seem worried about finding calculus procedures giving result 32, independently on their consistency with the context.*

Activities suitable for the classes	*1*	*2*	*3*	***4***	***5***	***1***	***2***	***3***	**Comments**

→
√ «How many black sots are there in Arabella's belt?»
«There are 10»
√ «What about white spots?»
«11»
«No, they are 22!»
√ The teacher suggests to write down a way to find the total number of spots. The following proposals are collected:

(a) 11 × 2 + 10
(b) 10 + 22
(c) 11 + 11 + 10
(d) 12 × 2 + 8
(e) 10 + 12 + 10
(f) 10 × 2 + 2
(g) 10 + 10 + 10 + 2

√ «Do you remember? We talked about this many times. The result does not let us understand how it was reached, whereas looking at the process we can see how we got to that particular product. For example … does 22 here show product or process in your opinion?»
«Product!»
√ «What should I do to show the process? How did we get 22?»
«We did 11 times 2!»
√ «You see? Now I understood that you *(addressing the author of (b))* did not calculate 7 plus 15 o any other way, but you did 11 times 2. If you tell me '11 times 2', you try to tell me exactly what you did in your brain. Now do this: write down on your notebook the process starting from 10 and getting to 22 so that you find the number of white spots. Then we will write it on the blackboard and we think about it »
The following writings are reported on the blackboard:

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→

(a) **10 + 12**
(b) **10 + 1 + 1 + 10**
(c) **10 + 1 × 2**
(d) **10 + 10 + 2**
(e) **10 + 8 + 4**
(f) **10 × 2 + 2**
(g) **11 + 11**
(h) **7 × 3 + 1**

√ «Good. Now let's see: who respected the rule I gave you? Which were the rules?»
«Starting from 10 was the first rule »
«And the second was to get to 22 »
Discussion leads to erase writings (g) and (h) because they do not follow the task, (a) and (e) because their authors' reasoning is not understandable. The following are kept:

(b) **10 + 1 + 1 + 10**
(c) **10 + 1 × 2**
(d) **10 + 10 + 2**
(f) **10 × 2 + 2**

√ (*to the author of (c)*) «can you show us where 10+1×2 is on the belt ?»
«black spots are 10, 1 because a line of white spots has one more than that with black spots, times 2 because there are two lines of white spots »
√ «Which operation should be done first? What must we do to make clear that that operation must be done first?»
«First we must do 10+1 and parentheses should be used!»
Parentheses are added up.
«It's an expression!»
«Doing 10×2+2, we saw 10 white spots in the two white lines and then we added the other two which were left »
√ «Which are the 'grown up' writings?»
«10 × 2 + 2 and (10 + 1) × 2» **26**

26. *This conclusion can become a chance to deal with the distributive law. Transition from*

10 × 2 + 2

to

(10 + 1) × 2

allows the teacher to deal with the so called ' extraction to common factor', which is more difficult to understand for pupils than the inverse.
In actual fact to understand the transition the first sentence should be seen as a slightly different paraphrase:

10 × 2 + 1 × 2

This representation allows recognition of the common factor '2' and therefore enables the pupil to highlight it in the distributive law, thus obtaining:

10 × 2 + 2 =
= 10 × 2 + 1 × 2 =
= (10 + 1) × 2

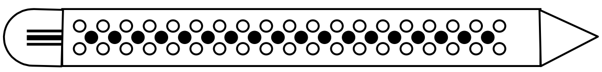
Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

8. Belts get longer

The activities go on proposing increasingly longer belts, searching for relationships between white and black spots. The aim is to make pupils approach a more and more expert use of a method for working hat highlights relational thinking. As concerns the search for regularities the most significant part of this phase starts off: that involving generalisation and formulation of a rule through making it explicit in both the natural and the formal linguistic codes.
As the diaries will make clear, the route might be long and arduous, sometimes dispersive, but anyway providing rich points for reflection, clarifications, deeper analyses, constantly in between creative exploration and search for logical rigour.

Diary 13 (5th grade, November)

A new coloured cardboard belt is shown.



√ «Gioacchino has a longer belt than Arabella's. Analyse and describe it»
«There are more spots»
«It has 3 parallel lines»
√ «But what is the important information?»
«the spots!»
«We need to count them»
«There are 19 black spots and…»
«40 white ones»
√ «OK. Listen. Carolina has a belt similar to Arabella's and Gioacchino's but it has 13 black spots. Do you think Carolina is thinner than Arabella??»
«She is fatter because her belt has more spots »
√ «Can you tell me how many white spots Caterina's belt has?»

→

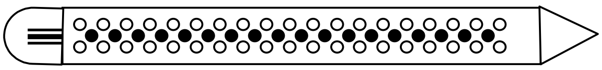
Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
The class is puzzled.
«20 altogether because if Carolina is not fatter than Arabella… I don't know …»
«They are 28, because both in Arabella's and in Gioacchino's belts white spots in a line were 1 more than the black one **27**. If the black line is made of 13 the white one will be made of 14, if there are just two lines it is 28.»

27. *An embryo of the first regularity was identified: the number of white spots in each line equals the number of white spots increased of 1.*

Diary 14 (5th grade, October)

The belt presented now is Gioacchino's, and this time there are 18 black spots.



√ «How many white spots are there?»
«I did double 19, I mean I added 1 to 18 and I did times 2 »
She goes to the blackboard to write it up:

2 × 19 = 38

«I know that white spots in a line are always 1 more than black ones, hence 1 plus 18 gives 19. Then I multiply 19 times 2 that is 38 and that is true for all belts »
√ «Good. I am not going to show you next belt: you need to imagine it. It is Obelix's belt: he is very fat therefore the belt has 40 black spots. How many white spots are there?»
«I do 40 plus1 and then 41 plus 41. Finally I find out that I have 82 white spots. »
A decision is made to represent these calculations through a very well known instrument for the class, arrow representation, which is very useful to favour the subsequent transition to the inversion of the law.
→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→

$$40 \xrightarrow{+1} 41 \xrightarrow{\times 2} 82$$

Pupils propose other correct procedures:

$$40 \xrightarrow{+1} 41 \xrightarrow{+41} 82$$

$$40 \xrightarrow{+40} 80 \xrightarrow{+1+1} 82$$

$$40 \xrightarrow{\times 2} 80 \xrightarrow{+2} 82$$

At this stage problems are solved immediately.

28

√ «Irene's belt has 12 black spots. How many white spots are there?»

«12 plus 1 gives13, which doubled gives 26. This is the number of white spots»

«Il did first a 10 times 2, then 3 times 2, then I added them up and the result is 26»

Others wrote:

$$12 \xrightarrow{\times 2} 24 \xrightarrow{+2} 26$$

✓ The teacher suggests to fill in a table with the data from the various belts:

Black spots	White spots
10	22
8	18
18	38
12	26

→

28. *The use of graphs turns out to be very powerful. In this case there are four equivalent graphs but only two of them, the first and the last one, can be generalised. In fact, the values of operators '+1', '×2' and '×2', '+2' are constant and independent on the initial value (in our case 40). In the other two graphs operators vary depending on the initial number, and this is an obstacle toward generalisation.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
It is decided to represent through graphs the way in which the number of white spots has been found.

$10 \xrightarrow{\times 2} \xrightarrow{+2} 22$

$8 \xrightarrow{\times 2} \xrightarrow{+2} 18$

$18 \xrightarrow{\times 2} \xrightarrow{+2} 38$

$12 \xrightarrow{\times 2} \xrightarrow{+2} 26$

Some pupils propose a different way of writing:

$10 \xrightarrow{+1} \xrightarrow{\times 2} 22$

$8 \xrightarrow{+1} \xrightarrow{\times 2} 18$

$18 \xrightarrow{+1} \xrightarrow{\times 2} 38$

$12 \xrightarrow{+1} \xrightarrow{\times 2} 26$

Two 'laws' for finding the number of white spots as a function of black spots are thus obtained. They write:

Black s. $\xrightarrow{+1} \xrightarrow{\times 2}$ white s.

black s. $\xrightarrow{\times 2} \xrightarrow{+2}$ white s.[29]

→

29. *Also in this case the important role of distributive law can be highlighted.*
After agreeing that:
n = number of black spots
b = number of white spots
pupils can represent the equivalence between the two writings in arrow form in algebraic language:

$$(n + 1) \times 2 = n \times 2 + 2$$

Now they must verify the equivalence. If they have already come across concepts described in Comment **26**, *they carry out the verification through a short chain of passages :*

$$(n + 1) \times 2 =$$
$$= n \times 2 + 1 \times 2 =$$
$$= n \times 2 + 2$$

The situation is favourable to help pupils understand how distributive law can be found also in 'unsuspected' situations ('seeing' in '2' the non canonical form '1 × 2' or '2 × 1'). It is stimulating for pupils to talk about a role of algebraic detective.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
√ A new problem is proposed, which is formulated in this way:

> Irma would like to buy a belt, similar to her friends' belts
> She knows the number of her white spots, i.e. 44 and would like to know how many black spots she has .

«They are 43»
«No, it's not true, because 44 are the white spots in both lines, hence they are 44 divided by 2 which is 22 for each line of whites, then I take out 1 and I find that there are 21 black spots »
«But we can also take 2 out of 44 and we get 42 and then divide by 2, getting 21»
«I did 44 minus 4 and that's 40, and then 40 divided by 2 which is 20, and finally I add 1» but he cannot explain the reason why he went on this way.
The class decides that there are two correct strategies:

44 : 2 = 22 and then 22 – 1 = 21

44 – 2 = 42 and then 42 : 2 = 21

Diary 15 (5th grade, October)

…
√ «Summing up, how do you calculate the total number of spots in a belt?»
«I multiply by 2 the lines of white spots and I add the number of black spots»
√ «How could you 'refresh' the first part of your sentence?»
«I multiply by 2 the number of white spots »
«I multiply the number of a line of white spots»
«I multiply by 2 the number of white spots of a line and I add the number of black spots » **30**

30. *It is important to work on refining language, also to make students used to a 'natural' and not shallow with mathematical terminology and, as successive step, with the conventional rules of written mathematical language. In this sense it must be made explicit that calculations are carried out on numbers and not on objects.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

Diary 16 (5th grade, October)

√ «Now let's see how we get to the number of white spots starting from that of black spots.»
Let's take five different belts of which only black spots are known.
Strategies elaborated by pupils are inserted in a table, like previous times.

P. neri	White s.	How did you do
10	22	(10 × 2) + 2
8	18	(8 + 1) × 2
18	38	(18 + 1) × 2
40	82	40 + 40 + 2
12	26	(12 × 2) + 2

Three different strategies have been used. Let us reflect on this point.
«Although different strategies are used the result is the same »
«Of course, because you actually do the same things: for instance, doing 40 + 40 is the same as doing 40 × 2 »
«And if I add 1 and then I double it, as in the second line, or rather I double it and I add 2, as in the first line, also this one is the same because it is as if I doubled also 1 » **31**
We imagine that Brioshi sends the following problem :

●	○
13	?
?	40

The first belt is not posing difficulties. Children find the number of white spots in two ways and recognise them as equivalent:

→

31. *Also in this case a discussion can be enacted on the distributive law, going back to aspects already described in Comments* **26** *and* **29**.
Starting from the pupil's remark, the following equivalence is written:

$$(8 + 1) \times 2 = 8 \times 2 + 2$$

and it is verified:

(8 + 1) × 2 =
= 8 × 2 + 1 × 2 =
= 8 × 2 + 2

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
White spots of Isabella's belt =
= (13 + 1) × 2 =
= (13 × 2) + 2 =
= 28

The second part of the problem, that about the belt with 40 white spots, activates a lively discussion.
Some say that there are 18 black spots, some others that they are 19.
A remark is made about the fact that apparently, inverting operations the result changes.
But what is the right answer then? **32**
«I drew the belt, starting from 18 black spots. I counted the white spots but they were 38. Then I added another black spot and in this way two white ones were added, and I got to 40. The right answer is 19»
«I made a drawing too, I saw that with 40 white spots I could do two lines of 20, therefore black spots became 19»
Everybody agrees on the result 19.

Note 4: Towards functions

The studies of correspondences are peculiar: they are defined in the set of natural numbers and they take natural values; moreover, for each natural number there is a *unique* correspondent. When this happens the correspondence is a *functional*-type one, or , more appropriately, a *function*.
Functions differ from one another by the properties they possibly have and therefore it is important to learn how to recognise them. In order to do this let us go back to Gioacchino's belt.
The functional correspondence

Number of black spots - number of white spots

Expressed by the law

$$b = (n + 1) \times 2$$

→

32. *'19' is the correct answer and represents the outcome of this reasoning: 'if I find the number of white spots in this way:*

$$b = (n + 1) \times 2$$

to find the number of black spots I will need to follow the inverse path:

$$n = b : 2 - 1$$

In the specific case, with b = 40:

$$40 : 2 - 1 = 19$$

The mistake of those who identify the answer '18' derives from the fact that they inverted operations, but kept the order in which they are carried out.
Their process to find b is correct:

$$b = n \times 2 + 2$$

but instead of inverting correctly the operation:

$$n = (b - 2) : 2$$

they developed a wrong inversion:

$$n = b : 2 - 2$$

In the specific case:

$$40 : 2 - 2 = 18.$$

The choice of a correct inverse formula can be a source for important discussions about the meaning of operations, the role of priorities in expressions and the use of parentheses.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→

must be seen as defined in natural numbers and with values in natural numbers.
This is the first remark to make pupils grasp the idea that *a formula alone is not enough to express a law, but fields of number in which it is considered must be pointed out*.

The correspondence has a particular property: to *distinct* numbers of black spots correspond *distinct* numbers of white spots, and this can be expressed in general terms saying that a function is *injective*.
But it does not happen that each natural number can be seen as correspondent of a certain number of black spots, because the number of white sots is always *even*, and this can be expressed in general saying that the function is *injective*: the set of correspondents is thus the set of even numbers.
This set, called *image* of the function, is the one we should refer to if we want to characterise the inverse correspondence.

Number of white spots-number of black spots

And this is possible due to the injectivity of the initial function.
Functional non-injective correspondences are not invertible.
It is clear that the inverse correspondence number of white spots- number of black spots must be defined on the set of even numbers, because it is not possible to have an odd number of white spots in the belts we are exploring.
Taking into account the formal expression of this inverse correspondence

$$n = b : 2 - 1$$

clearly it must not be meant for any natural number, but only for even numbers, i.e. for numbers n which can be represented in terms of 2m.

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
These reflections, suitable for lower secondary school, constitute an experiential background for basic mathematical concepts such as:

- function,
- domain of a function,
- image of a function,
- injective function,
- surjective function,
- biunivocal function

and for conceptualisation of important mathematical facts, such as the identification of conditions under which a correspondence is invertible (as we saw, this must be at least an injective function).

9. A second field of search for regularities: Christmas decorations

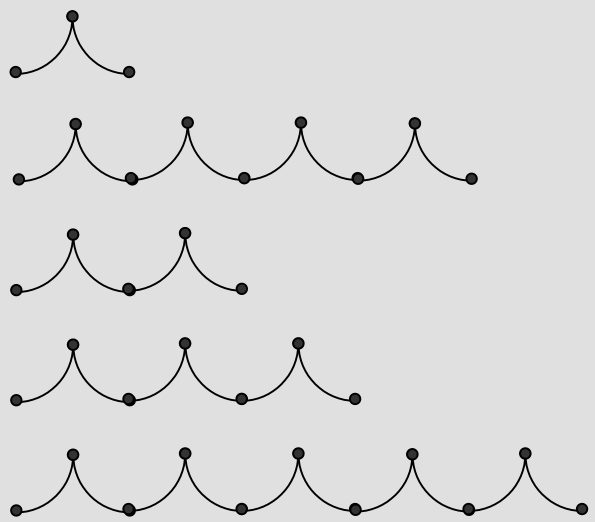
An extremely rich resource for search for regularities is represented by friezes, drawings, frames, festoons and embroideries.
The next proposal is an example from this area. It is a story the teacher tells while showing purposefully prepared material.

Alice prepares some wall Christmas decorations with fir branches and coloured balls.
With branches she constructs many festoons like this one:

And then joins them together decorating them with balls and making decorations with various lengths.
When her older sister Caterina goes back home and sees some ready decorations, she notices that they all share something, although they have different dimensions.
→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→



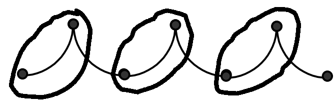
"Beautiful!" Caterina says. Then, since she gets very curious when she sees situations like that, she wonders: 'Which relationship is there between the number of festoons and that of decorations?'

Diary 17 (4th and 5th grades together, February)

A discussion within a group of pupils emerges.
«In one festoon there are two balls!»
«No, in one festoon there are three balls, hence I multiply by 3 » **33**
«I think there are many pairs of balls and in the end there is always one left » **34**
√ «Is it possible to have a festoon with 800 balls?» **35**
«I think they should be 801 » **36**
«You can't because there are pairs and then plus 1 » **37**
«In the end there is always one ball more»
«The number of balls will always be odd, due to that plus 1 » **38**
«Yes, it must be odd, always» **39**

33. *The pupil applies a multiplicative model without thinking.*

34. *The pupil 'sees' the decoration in this way :*



35. *There is a remarkable jump in the question. Pupils must leave exploration of small cases and generalise thus forcing their mental images. The request may appear not justified in this moment but it is up to the teacher to evacuate how suitable this can be, depending on the situation (raised hands, willingness to intervene etc.).*

36. *The pupil shares the previous statement but has no correct perception of the model yet.*

37. *The pupil has an appropriate view which, once share, will lead the class to the problem's solution.*

38. *These pupils have this type of perception:*

They underline the odd nature of 'final' balls but not the fact that they are the double plus one with respect to festoons.

39. *When interventions are so heterogeneous it is appropriate for the teacher to reinforce conclusions, perhaps going back to the pupil who proposed the correct view of the situation.*

Activities suitable for the classes	*1*	*2*	*3*	***4***	***5***	***1***	***2***	***3***	**Comments**

Diary 18 (4th grade, February)

√ «Make a decoration with 11 festoons and count how many balls are in there »

«The balls are 23»

√ «And now: how can you find the number of balls if there are 125 festoons? Would you do a drawing here too? It would be too long, wouldn't it? Think about it and write down how you understood. Then we will do this: I write a YES column and a NO column on the black-board.

YES	**NO**

Then I will ask you how many balls you found. I will write each number in one of the columns, depending on whether it is right or wrong. No-body will be allowed to ask anything. You will need to try and understand on your own why that certain number is right and the others are not. Let's start »

The first values are spoken up. They are all wrong. Some of them are so close to the cor-rect one that a third column, the 'YO' one is in-serted exceptionally.

YES	**YO**	**NO**
	250	**375**
	252	**625**
		137

Another 'YO' result is proposed and finally the correct one.

YES	**YO**	**NO**
251	**250**	**375**
	252	**625**
	253	**137**

√ «How did you find 375?»

«If a festoon has 3 balls I do 125 × 3 and I find the number of balls»

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
«I did this too. If each festoon has 3 balls I do 125 times 3» **40**
«With 125 festoons I draw 252 balls: for each upper ball there is a bottom one: 125 plus 125 plus the two at the sides » **41**
«The difference between 11 and 23 (she refers to the decoration with 11 festoons) is 12 : 125 + 12 = 137» **42**
«I multiply the number of festoons by the two lines of balls»
«With 125 festoons I draw 251 balls. Number of festoons times 2 plus 1 because there is one ball less on the top»

Diary 19 (5th grade, February)

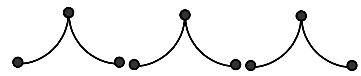
√ «Let’s find a way to express the number of balls through that of festoons»
Some pupils are putting the number of upper balls in relationship with that of bottom balls. We need to make them think about what they are looking for again. **43**
√ «Now you have become export in using tables and you could construct a table and insert the numbers from Filippo’s drawings, if that can help (*Filippo is a classmate who intervened earlier*). Then look at what happens»
«Shall we put the number of festoons or rather the number of balls?»
«Both!»
√ «The table is the compass that allows you to get your bearings in the ocean of regularities. Last time I asked you to construct your own compass. Some of you found useful ones, others less powerful ones. I copy down Swann’s compass, which is very clean»

1	**2**	**3**	**4**
1 dec.	**4 dec.**	**5 dec.**	**3 dec.**
3 balls	**9 balls**	**11 balls**	**7 balls**

The number of balls is always twice the number of festoons + 1.

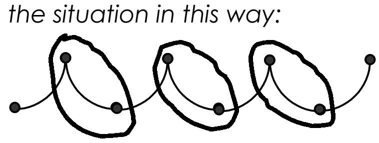
→

40. *the mistake comes from the fact that the pupil repeats the first decoration ‘in toto’, thus getting a series of festoons like the following:*



Actually in this version each wave corresponds to three dots.

41. *the pupil – independently on the number of waves – perceives the situation in this way:*



42. *In the pupil additive thinking seems to have not been influenced by proportional thinking.*

43. *This view maybe goes with the experience of exploring belts. It is appropriate for the teacher to g back to these ideas, which are initially distracting, so that pupils can be shown different ways of tackling the same situation. In this case pupils can be led to conceive the unifying of the two correspondences through identification between vertex of a festoon and decoration.*

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
√ «How would you translate the sentence under the table for Brioshi?»
«P = … don't know…»
√ The teacher copies down Swann's sentence, changing it slightly and then highlights some groups of words.
The class observes quietly.

The number of balls is one more
Than double the number of festoons

√ «Now translate each group of words or each word in mathematical language. Let's agree on calling the number of balls …»
«p!»
√ «… and that of festoons…»
«f!»
√ «very good! And now let's translate»
The following translations are written:

The number of balls is one more
p = 1 +
than double the number of festoons
2 × f [44]

We write it again in a compact form:

$$p = 1 + 2 \times f$$

√ «And this is the translation for Brioshi!»

44. *The strategy on the blackboard, consisting of favouring the production of an algebraic writing as a* literal translation *of a correspondent one formulated in natural language- is very productive with both young and older pupils, also in 8th grade.*
Natural language – in the theory of algebraic stuttering- represents the main didactical mediator to enable an understanding of semantic and syntactic of algebraic language as regards both production, as in this case, and interpretation.
Formal language – in the initial phases of its construction – should rise from a process of translation, *educating pupils to grasp analogies between the syntactic structures of the two languages, in this case: to see 'p' as the subject of the mathematical sentence, '=' as copula of nominal predicate, synonymous of 'it's equal', and so on.*
During these initial steps we gradually construct in pupils the aware respect for rules that support a correct construction of algebraic writings.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

Expansion 3: Correspondences are analysed in depth

In the case of festoons and decorations, likewise that of the function coming from the exploration of belts, not each natural number can be seen as number of balls corresponding to a certain number of festoons: in actual fact the number of balls is always *odd*.
In case we want to push the analysis forward to the inverse correspondence in its formal representation, pupils need to be led to reflect on the fact that the formula obtained from p = 1 + 2 × f, that is

$$f = (p - 1) : 2$$

is to be considered only for the set of *odd numbers*. Only in this case the formula does fit with the situation because you get a natural number as correspondent. In fact, if p is an even number, you should get an odd number by subtracting 1 and the division by 2 would be a decimal number.
These thoughts are important to underline that a formula is not to be considered alone, but always in relation to the number sets it refers to. This will help to clarify an epistemological misconcept, quite spread among students: that a formula represents a law, that it is a *function*. It is appropriate to make pupils conceptualise a formula as an open sentence, whose meaning and character depend on the number environments it refers to.
As regards syntactic aspects, a interesting expansion can be the comparison between correspondences' formulae <number of black spots – number of white spots> in the case of belts and <number of festoons – number of balls >, in the examined case; in other words between

$$b = (n + 1) \times 2 \qquad p = f \times 2 + 1$$ [45]

An important thing will be the promotion of a discussion about pupils' perception of formulae, the explicit expression of processes they represent and the priority of steps to be taken when formulae are inverted

45. *For example, before dealing with inversion, one can distinguish procedural descriptions like ' what I do' :*

'to find b I must add 1 to n and then m multiply by 2'
'To find p I multiply f by 2 and then add 1'

from those saying 'what is' the mathematical object:

'b is the product of the sum of n and 1 and 2'
'p is the sum of twice f and 1'.

Another conclusion might be the observation of the fact that b is certainly an even number and p an odd number.

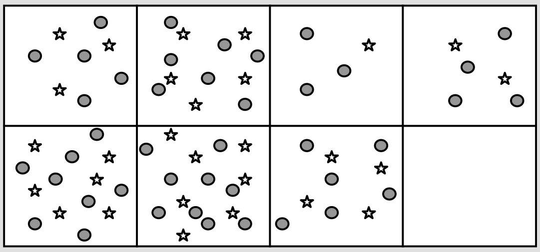
Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

10. Stories about tidy children

Besides friezes interesting hints are given by sets (for instance collections) made of two different types of objects linked by a 'law' that is easy to be expressed (for example, that the number of objects A is twice those of B, or that the number of As is due more than Bs and so on).
Classes deal with situations in which, for instance, they must help some children to tide up their toys and collections.
An example will illustrate these situations:

Domenica's collection

Domenica, a very tidy child, is dealing with her collection of sea stars and sea chestnuts.
She decided to reorganise it following a secret rule. **46**



Find the rule that Domenica uses to tide the box up.

Diary 20 (4th grade, January)

«In each box there is an even number with both elements between 4 and 16 ».**47**
«In each box there is always either an even or an odd number both for sea stars and for sea chestnuts » **48**

46. *A panel portraying the collection, similar to the drawing, can be shown. The class must find out that Domenica's rule keeps the difference between the number of sea chestnuts and that of sea stars constant in the various parts of the box.*

47. *She means that in each box there is an even number of objects and that this number varies from a minimum of 4 to a maximum of 16.*
With older or more expert pupils we might get to justify the pupil's remark through a reading of the situation in terms of 'sum of two odd numbers' and 'sum of two even numbers' that are anyway even.

48. *The pupil means that in the different parts the number of sea stars or sea chestnuts are both odd or both even. The remark might be linked to that of the previous classmate.*

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

Diary 21 (5th grade, January)

«I counted objects in the first box. Then I see two sea stars less than sea chestnuts. But…»
«In each box stars are less and chestnuts are more»
√ «Write down the rule in your workbook»
Pupils work individually. Then the teacher collects protocols and classifies answers. [49]
√ «You did a very interesting work. I divided your answers in two groups. Now I write them at the blackboard»

1st group (6 pupils)
'Domenica puts in each box
2 chestnuts more than stars'

2nd group (5 pupils)
'Domenica puts in each box
2 stars less than chestnuts '

√ «What do you think?»
«The meaning is always the same. The sentences complete each other if they are joined together»
√ «It is exactly what the 3rd group says. I am going to write it now»

3rd group
'Chestnuts are 2 more than stars
and stars are 2 less than chestnuts'

√ «But then is the first sentence enough, or the second, or do you need both?»
«One is enough because it is logical that if chestnuts are 2 more, stars are 2 less»
√ «What do you think of this sentence by Federica?»

Federica
'We must 2 out of chestnuts and we find the rule'

→

49. *In this phase it might be useful that pupils have the worksheet with the situation on it, so that the teacher may carry out a fine analysis of protocols also after activity in the classroom.*

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
«It's not enough because in this way chestnuts equal stars»
The three chosen sentences are written on the blackboard:

a) In each box there are 2 chestnuts more than stars
b) In each box there are two stars less than chestnuts
c) You must take two out of chestnuts and you find the rule

√ «How can we modify the text of the first group to make it more understandable and correct from the mathematical point of view?»
The class is puzzled.
√ «Are 'chestnuts' two more…?»
«It is the number of chestnuts which is two more than the number of stars »
√ «One of you proposed a representation of the rule through mathematical language. Try to write it as if you should send it to Brioshi»
Pupils' proposals are transcribed on the blackboard:

(a)	**C = S + 2**	**S = - 2**	
	S = + 2	**C= 4**	
(b)	**C = S + 2**	**S = C – 2**	**(6 pupils)**
(c)	**S = (- 2)**	**C= + 2**	
(d)	**S.S. – 2**	**S.C. + 2**	
(e)	**C = 10 -2S = 8S**		
(f)	**C = 2 + S**	**S = 2 – C**	
(g)	**A = 2 more than B**	**B = 2 less than A**	
(h)	**2 more than stars**	**2 less than chestnuts**	
(i)	**C= + 2 than S**	**S = - 2 than C**	

The debate starts.
«For (f) : where there is S = 2 – C the number of chestnuts must be either 1 or 2 because if it is higher you cannot subtract. This representation is not good»
→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
√ «Comparing (b) and (f) what can you say?»
«If S + 2 holds then also 2 + S does. It is the commutative law. But it does not hold for subtraction»
√ «So I can take (f) out. Now look at (g) and (h) and (i)»
«They used a mixture of natural and mathematical language»
√ «It is as if we went to England not speaking English very well. If we must eat we can use a mixture of signs and some English words. Although we have not used the language well we reach our purpose, but certainly we can find many who do not understand us»
«I wrote (e), but I wanted to use number 10 to make an example »
«What did you want to represent with 10?»
« 10 sea chestnuts»
«So if you take 2 sea stars out of 10 chestnuts, how can you get 8 stars?»
«In this way you say that the number of chestnuts equals 8 stars. It's no good»
Discussion goes on until children decide to choose rule (b).

As we have already pointed out elsewhere, it is important that pupils deal with the problem of translating between two languages.

Diary 22 (4th and 5th grades together, January)

Each pupil writes the rule on the workbook. Some sentences are written at the blackboard:

(1) In each box there are always two chestnuts more;
(2)In each box there are two sea stars less than sea chestnuts;
(3) Domenica always adds for stars and for chestnuts one or two numbers more

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
√ «Translate these sentences in mathematical language »
Some typical answers are reported on the blackboard:

	Num. of chestnuts	**num. of stars**
(a)	**2 + a**	**2 – a**
(b)	**s + 2 = c**	**c– 2 = s**
(c)	**d + 2 = c**	**a – 2 = b**
(d)	**b + 2 = a**	**a – 2 = b**

√ «What do you think about translations?»
«It's better to keep (b), because it helps to understand stars and chestnuts »
√ «Can you find another way to express the rule? What does that 2 represent?»
«It's the difference»
«It's the number of chestnuts»
«2 is the number you take out and you add »
«2 is the difference between the number of chestnuts and the number of stars »
The class agrees on the correctness of the last definition.
√ «Translate it into mathematical language »
Some particular sentences are proposed and transcribed:

2 is the difference between the number of chestnuts and the number of stars

3 – 1 = 2   2 = 4 – 2

√ «But in these two writings what are '3' and '4'?»
«The number of chestnuts!»
√ «Good! What about '1' and '2'?»
«The number of stars!»

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

→
√ «How can you translate the sentence written in Italian?»
«2 equals c minus s»
√ «Excellent. Let's write it down. Can you write it in other ways?» **50**
The following sentences are dictated:

2 = c – s   **s + 2 = c**   **c– 2 = s;**

Concluding pupils point out that they are three different ways to say the same thing, three equivalent ways. **51**

50. *Classes involved in this diary,- especially the 5th grade- have already carried out this type of activity. We deem useful for the 4th grade- although inexperienced- to look at the construction of these representations because they are elaborated within a collective discussion (we recall here the ArAl theory of algebraic stuttering). Moreover they do not aim at achieving manipulative competence, which is completely stranger to the work presented in these pages.*

51. *To underline the linguistic aspect of the activity, it might be fruitful to reflect with pupils on the fact that the three representations are similar to those that we call* paraphrases *in the study of language.*

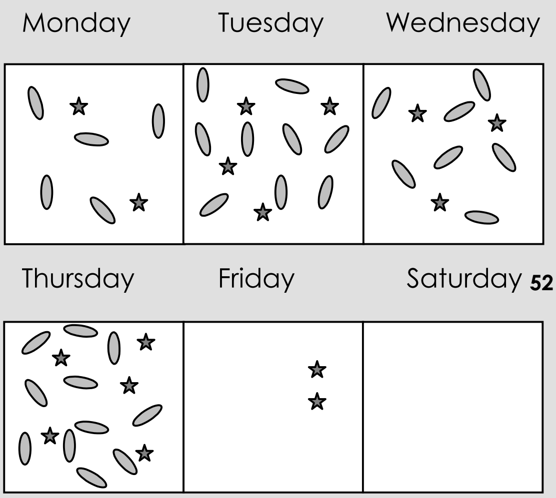
Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

11. Biscuits for breakfast

An interesting problem, analogous to that of sea chestnuts and sea stars, presents one further difficulty: a cell contains only one element and another one is empty.
The teacher shows a poster illustrating the situation and presents it.

Chocolate shortbreads and sponge fingers

Clotilde likes chocolate shortbreads and sponge fingers.
She eats them daily from Monday to Saturday, each time in different quantities, but following a rule she gave to herself.
For Friday and Saturday only chocolate shortbreads eaten by Clotilde are represented.



Once the introduction and related clarifications are over, the teacher asks pupils to reflect on this question:
Can you find this rule?
Then she assigns the first task:
(1) Write down Clotilde's rule in natural language.

52. *The empty Saturday cell represents an undoubtedly difficult element for pupils: apparently the message is that Clotilde does not eat either shortbreads or sponge fingers on that day. Exploration and achievement of the rule will lead pupils to recognise that Clotilde* eats only one sponge finger, *but the prediction of this conclusion conflicts with commonsense. Moreover it is the consequence of an important achievement on the plane of generalisation.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

Diary 23 (5th grade, January)

5 pupils out of 13 do not write anything.
√ The other sentences are quickly classified by the teacher and the most representative ones are transcribed at the blackboard:

(a) Stefano takes from sponge fingers the same number of shortbreads, multiplies it by 2 and adds 1 ***(3 pupils out of 13)***
(b) She eats an odd number of sponge fingers and shortbreads between 1 and 5.
(c) Shortbreads are always one more than the others.
(d) One day she eats a bigger quantity and the other day a smaller quantity.

√ «Make your own remarks on these rules»
All pupils after a collective discussion, agree on the choice of (a).
√ «Try to write (a) in other ways»
We transcribe:

(a_1) The number of sponge fingers is 1 more than twice the number of shortbreads.
(a_2) The number of sponge fingers is twice plus 1 the number of shortbreads.
(a_3) The number of shortbreads multiplied by 2, adding 1, is equal to the number of sponge fingers.

√ «Very good. Now translate The same sentence in mathematical language » **53**

53. The situation under discussion is very rich and concerns both the name of the quantities under consideration and the priorities of operators +1 and ×2.

(a)	**1 × 2**	
(b)	**a.+ 1 × 2**	**a = number of sponge fingers**
(c)	**sf + 1 × 2**	
(d)	**a × 2 + 1**	
(e)	**sf + 1 × 2 = a**	
(f)	**sf = sh+ 1 × 2**	
(g)	**a = b × 2 + 1**	
(h)	**a × 2 + 1 = b**	**a = number of shortbreads**
(i)	**(a – 1) × 2** **(Silvia)**	

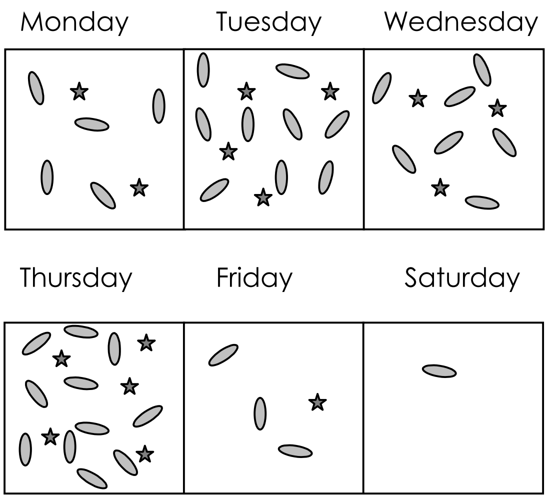
→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
«Silvia meant to indicate the number of shortbreads with 'a', but she made a mistake because she must write divided by 2, that is:
(b – 1) : 2, she did the converse reasoning»
After a collective analysis of the writings, the following rule is chosen:

$$b = a \times 2 + 1$$

√ «Can you understand what happens is there is no shortbread?»
The class immediately uses the formula and finds that there is only one sponge finger. Many are puzzled by the situation, which is intuitive on a mathematical plane, but not supported by an equally strong concrete image.
Pupils complete the drawings of biscuits on Fridays and Saturdays.



Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

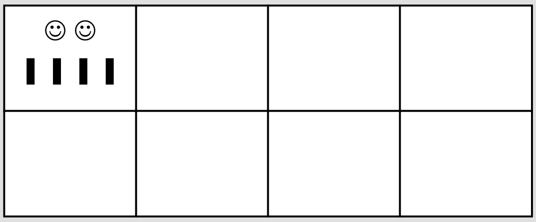
12. Gioacchino's box of toys

An analogous situation to Clotilde's with more complex problematic aspects.

Diary 24 (5th grade, December)

√ A magnetic board is hung on the wall and the following situation is presented:

Gioacchino is preparing a game for her sister Francesca.
He divided the magnetic board which is hanging on their bedroom's wall in 8 equal rectangular spaces, then he stuck in the first space two magnetic puppets and four Geomags.



Gioacchino wants Francesca to find out the link he set up between the number of puppets and that of Geomag elements and that she puts in other spaces different numbers of the same objects, respecting the same rule.
Put yourself in Francesca's shoes and try to find out the Rule. **54**

√ «I propose you start in this way: in your opinion how many puppets and Geomags will Francesca put in the second space?»
Pupils copy down the drawing on their workbooks and represent puppets with a circle and Geomags with a cross. Then they start reasoning.
After a while the discussion is started.
«I think she adds one piece in each part and therefore she will put: OOO ×××××»
«To me she adds a puppet and takes a Geomag out and she gets: OOO ×××»

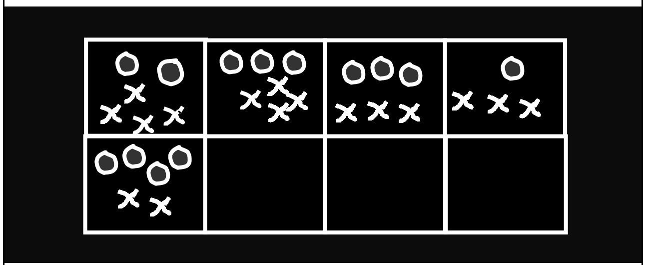
→

54. *The number of objects (2 puppets and 4 Geomags) have been chosen because they enable different interpretations of the situation, thus making it possible for pupils to refer to both operators +2 and x2.*
In this way Gioacchino's law won't probably be identified immediately, but pupils will get used to hypothesising relationships and comparing them with that of our protagonist.

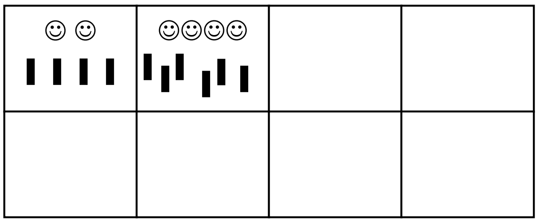
Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
«She takes one piece of each type from each part and gets O ×××»
«In my opinion she adds 2 puppets and takes 2 Geomags out, getting OOOO ××»
Pupils come to draw their proposals on the blackboard. **55**

__55.__ Pupils formulated their hypotheses freely, as it is necessary in the development of algebraic stuttering. But now it is time to stop and think, and understand that many conjectures are not consistent with the proposed situation.



√ «Now let's see whether some of you found the correct rule. Look. Since Francesca is not able to find it, she asked her brother to add puppets and Geomags in the second cell. Here is what Gioacchino added:»



A comparison is made with the drawings on the blackboard.
√ «Let's see… no, I don't think anyone did the same as Gioacchino… and so, what could Francesca put in the third cell?»
«We increased puppets of 2 and Geomags of 2 moving from the first to the second cell, so we will do the same moving from the second to the third » **56**
√ «Is this your proposal? Let's see whether this is the same as Gioacchino's…»
The magnetic board is updated:

→

__56.__ These pupils are focused on the sequence of the number of puppets in the cells (2 in the first cell, 4 in the second), they hypothesised that there are 6 puppets and 8 Geomags (2 more) in the third and they do not see the relationship between the two groups of objects within each cell.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→

The class is bewildered and starts getting angry with Gioacchino.

Enrico finally hypothesises that there is a link between puppets and Geomags within each cell, but his reflection is not grasped by others.

Some ask that Gioacchino adds pieces in other cells too.

The board is further updated little by little, leaving pupils the time to reflect:

The class continues the search for the law, but they do not manage to find it.

Finally Enrico is given consideration:

« Geomags are always 2 more than puppets!»

The class verifies that Enrico is actually right and they congratulate him.

Next Diary deals with a problem with a slightly different form but identical substance. The story is told without any support from concrete models or drawings.

The educationally interesting aspect is that – differently to the class in the previous Diary- this class does not work at iconic level- making and comparing drawings- but makes a massive use of the Yes and No Game (see Note 2, 'The main instruments').

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

Diary 25 (4th and 5th grades together, March)

√ The following situation is proposed: **57**

Gioacchino has some boxes in which he puts his Geomags and magnetic faces.
His parents, entering his room see that in the first box he put 2 Geomags e 4 faces.
Since they know their son's taste for tidiness they wonder how many Geomags and faces might be in the second, in the third box and so on.
In other words they try to understand how Gioacchino is organising his boxes.
Put yourself in Gioacchino's parents' shoes.
Explain your hypotheses in your workbook.

√ «When you are ready we will write your proposals on the blackboard».
A huge amount of ideas come from pupils:

4 G	**6 F**
3 G	**5 F**
4 G	**8 F**
5 G	**3 F**
8 G	**16 F**
1 G	**7 F**
6 G	**3 F**
3 G	**6 F**
8 G	**9 F**

√ «Listen, to understand which are good and which are not we will use the Yes and No Game and you will have to try to understand how things work on your own».
The list on the blackboard is updated:

→

57. *That proposal is a trace of presentation, as usual. It is not meant to be a worksheet to be given to pupils.*
It can be told by the teacher in his/her own words or rather read aloud with interruptions, comments clarifications. Of course it can also be a written text to assign to pupils.
What matters is that when the class starts working, they must have clearly understood the sense of the task and there should not be interferences blocking the exploration of the situation.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→

YES			NO
✕	4 G	6 F	
✕	3 G	5 F	
	4 G	8 F	✕
	5 G	3 F	✕
	8 G	16 F	✕
	1 G	7 F	✕
	6 G	3 F	✕
	3 G	6 F	✕
	8 G	9 F	✕

The class does not know what to do.
√ «Listen: I challenge you: in the 4th box Gioacchino put only 1 Geomag. How many faces would you put?»
«3 F»
«2 F»
«6 F»
«8 F»
√ The teacher makes two other yes and no columns and inserts pupils' answers:

YES			NO	YES	NO
✕	4 G	6 F		3F	
✕	3 G	5 F			2F
	4 G	8 F	✕		6F
	5 G	3 F	✕		8F
	8 G	16 F	✕		
	1 G	7 F	✕		
	6 G	3 F	✕		
	3 G	6 F	✕		
	8 G	9 F	✕		

√ L'insegnante prepara una tabella con le coppie corrette scoperte sinora «Vediamo un po'… nella 5a scatola metto 7 Geomag…

→

Activities suitable for the classes	*1*	2	3	***4***	***5***	***1***	***2***	***3***	**Comments**

→

YES	**NO**
2 G	**4 F**
4 G	**6 F**
3 G	**5 F**
1 G	**3 F**
7 G	**?**

… How many faces would you put?»
Four wrong answers are given before the correct one:

YES	**NO**
9 F	**10 F**
	4 F
	5 F
	8 F

√ The question mark is substituted by 9F:

YES	**NO**
2 G	**4 F**
4 G	**6 F**
3 G	**5 F**
1 G	**3 F**
7 G	**9 F**

After few seconds the class understands. Many hands are raised, asking to intervene.
√ «Oh, good! Tell me in Italian language the rule that Gioacchino most probably follows »
«He always adds 2 faces and takes 2 Geomag out»
«He always adds 2 to the number of Geomags»
«He always adds 4 faces more and one Geomag less»
«You need to add the same number of faces as Geomags, plus two more faces»

→

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
«You add 2 faces with respect to Geomags»
√ «What message would you send to Brioshi? Use mathematical language»
The teacher inserts again pupils' proposals in YES and NO columns.

YES	NO
G+ 2 = F	2G + 2F + 2F
F = G + 2	G + 2 – 1 = F
F – 2 = G	7G + 3F + 2 = 12
G = F - 2	F + 2 = G
2 + G = F	+ 2F – 1G

√ «Look at the YES column. How many different ways to write the same thing! You see? We say these writings are … »
«Equivalent!»
In the subsequent discussion pupils point out that the commutative law was applied and that equal things are written to the right and the left of the equal sign.
√ «Irene had proposed earlier: 'If I put 7 faces, how many Geomags will I need to insert?'. What would you answer?»
«5!»
√ «Now: if I start from the number of Geomags …»
«You add 2!»
√ «And if I start from the number of faces …»
«You take out 2!»
«O Geomag… »
√ «Hold on! We must be clear! Express the rule in Italian language, while looking at those written in mathematical language in the YES column »
«The number of faces equals the number of Geomags plus two »
«To find the number of faces you must add 2 to the number of Geomags »
√ «Good: the first definition says what the number is; the second describes *what you do* to find it. We will come back to this later » **(see Note 5)**

Note 5: 'Being' and 'doing'

The two definitions recall a crucial point.
The second definition:

'To find the number of faces you must add 2 to the number of Geomags'

is the description of the process (to find … you must add). This links to the 'doing' universe, it is an operative description.
The first, instead:

'The number of faces equals the number of Geomags plus 2'

is the description of the object. It links to the 'being' universe, it is an ontological description, expressing the relationship 'being equal' between the two numbers.
The first lies at a cognitive, factual level and thus more concrete and direct; the second lies at a metacognitive level since it represents a process of getting distanced from the procedure and therefore a conceptually higher point of view.
In an algebraic stuttering view, we might say that the first definition reflects an arithmetic mentality ('to find' points to an implicit conception of '<u>equal</u>' as 'directional operator' and explicitly recalls 'doing a calculation'.
The second, instead, can be reduced to an 'algebraic' conception, since 'it is equal' reflects a view of equivalence and prepares the round for its translation into formal 'F = G + 2'.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

Third phase

13. From Gioacchino to Brioshi

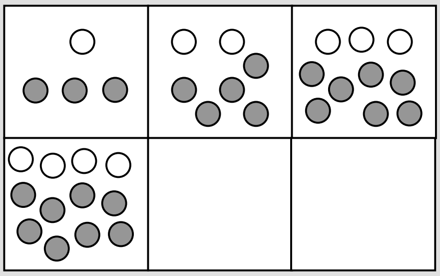
As we already pointed out elsewhere, both in this and other Units, linguistic aspects are very important, also in activities concerning the search for regularities, mainly as concerns the move from one to another linguistic code, and hence the interlacing of translations between iconic, natural and mathematical language.
Brioshi acquires, in this context, relevant didactical importance, obliging pupils to syntactic correctness in the use of formal language. Brioshi, once introduced in the class, becomes a very familiar figure, and his interventions ("But do you think Brioshi understands this thing you wrote?") make pupils understand quickly the terms of the 'linguistic quarrel' opened up by the teacher in that particular situation.
Sometimes Brioshi plays a protagonist role, mainly with young pupils and the exchange of messages with him can become the actual ground on which some teaching activities are set up.
We deem important to propose some Diaries illustrating this kind of episodes.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

Diary 26 (5th grade, January)

√ The following situation is proposed:

Gioacchino put some red and yellow marbles in equal boxes and left some boxes empty. **58**



He stuck a label on top of the boxes:

r = g + 3

Then he put two heaps of marbles in two containers: the red ones in one and the yellow ones in the other .
He organised a sort of toll for his friends: when they enter his room, before starting to play with him, they must read the label and put some red and yellow marbles in one of the empty boxes, taking them from containers, so that his rule is respected.

Pupils have already deduced formulae and thought about them, and therefore they have no difficulty to understand the situation. The know that ‘r’ stands for ‘number of red marbles’ and that ‘y’ stands for ‘number of yellow marbles’. **59** The draw the box of toys in their workbook and the put marbles inside.
Data found by pupils are then inserted in a table:

→

__58.__ The white circle indicates a yellow marble and the grey circle indicates a red marble.

__59.__ It often happens that during discussion the teacher realises that pupils are making an ‘inversion mistake’ and interpret the formula as if it were written this way:

(i) g = r + 3.

Should this situation occur, one might initially propose, instead of the writing:

(ii) r = g + 3

an equivalent writing:

(iii) g + 3 = r

The advantage for pupils would be that in initial phases of construction of algebraic stuttering, they might recognise in (iii) a reassuring similarity with a sentence like:

5 + 3 = 8.

The equality symbol would still keep a strong character of directional operator and there would still be something to the right recalling ‘the result’.
It must be very clear though, that this phase represents only a sort of temporary picklock towards the necessary acquisition of more evolved knowledge, as requested in this case.

Activities suitable for the classes	*1*	*2*	*3*	***4***	***5***	***1***	***2***	***3***	**Comments**

→

Yellow marbles	**Red marbles**
1	**4**
2	**5**
3	**6**
4	**7**
…	**…**

√ To start pupils are asked to write the rule for Brioshi in different but equivalent ways.
Proposals are:

$y + 3 = r$ $\qquad r - 3 = y$

Diary 27 (5th grade, January)

√ We imagine that Brioshi sends a message in which he supposes Gioacchino's rule to be the following:

$$r + y = 10$$

The class interprets the rule easily. They start thinking about the possible values. As values are proposed they are written at the black-board:

r	**5**	**1**	**7**	**6**	**8**	**4**	**0**	**10**	**2**	**3**	**9**
y	**5**	**9**	**3**	**4**	**2**	**6**	**10**	**0**	**8**	**7**	**1**

√ «What can you observe? Write it down on your workbook and we will read it later»

(a) In each pair the sum of numbers equals 10
(b) In each row there are all numbers between 0 and 10

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
(c) Pairs are repeated but swapped: for each pair there is a symmetrical one, except 5-5 which stays the same even if they are swapped **60**
(d) Pairs are made of two numbers which are either both even or both odd
(e) There are 11 possible pairs

√ «What if we wanted to write the same rule in a different way? »
The following proposals are written on the blackboard:

(a) $10 = r + y$
(b) $y + r = 10$
(c) $10 – y = r$
(d) $10 – r = y$
(e) $r = 10 – y$
(f) $y = 10 – r$

√ «Since you are so good I will show you another and more difficult message from Brioshi. Let's see if you can interpret it»

$$r + (2 \times y) = 15$$

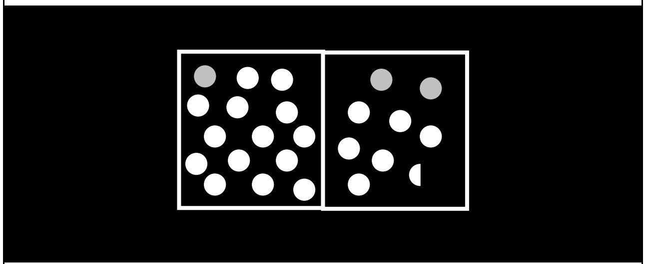
«It is better if 'r' is odd». He cannot justify this intuition.
«I substitute any odd number for 'y', then I take the result of 2 times y and I add a number to it so that I get to 15 »
Both pupils who intervened start from twice a number y and add r to it.
√ Pupils are suggested to draw Gioacchino's boxes and fill them in as correct pairs of values are identified.
Many pupils find some correct solutions for the equation, but then they do not represent the pair (r;y) in the drawing, but rather the pair (r;2y). For example, if they have found the solution 1;7 they draw 1 red marble and 14 yellow marbles. **61**
→

60. *This can be an opportunity to reflect on the commutative law for addition.*

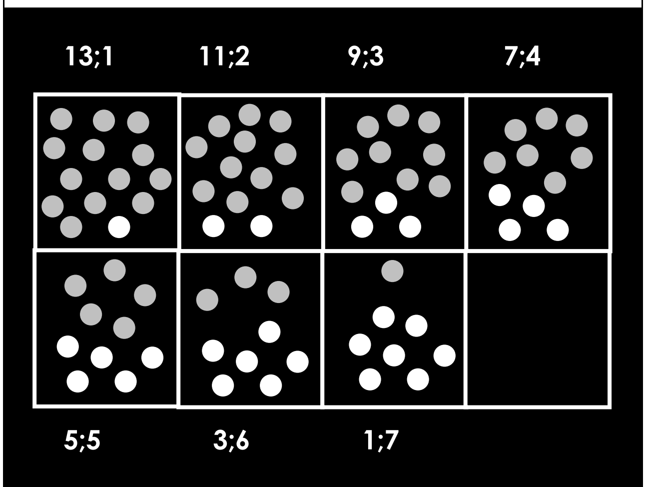
61. *Naive mistakes in these initial phases are understandable. Once they found the product between 2 and 7, the number 14 is 'condensed', whereas 7 is 'evaporated'.*
In so doing, 'y' is not seen as the variable on which one intervenes with the operation, but rather it becomes one thing with its double.
Discussion will clarify this aspect. It is another meaningful episode among those that can occur in activities centred around the construction of algebraic babbling.

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
A pupil does his drawing and then justifies it «I put 2 red marbles, then I need 13 to get to 15 and 13 divided by 2 is 6.5 and thus I drew the half marble. » **62**



Pupils find seven solutions, represented sequentially on the blackboard. They guess that one is missing but they cannot identify it. **63**



The mystery is finally uncovered:
«In the last box there is no yellow marble and there are 15 red marbles! 0 times 2 is 0 and then 15 is left!»
The table is completed.

62. *The teacher can use this opportunity to clarify that the reference number environment is that of natural numbers, although the formula can be easily interpreted in other numerical contexts.*

63. *The difficulty stays in the fact that pupils always think about non null numbers.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

14. Message exchange with Brioshi [64]

The environment for the exchange is that of Gioacchino's boxes. The class in the next Diaries is always the same.

Diary 28 (5th grade , January)

Today is the day. Finally the first 'real' message by Brioshi has arrived: [65]

ベッルーノの親愛なる友達へ
この問題に答えてください。

$$\text{Ж} = (\& \times 3) - 1$$

それではまた。

BRIOSHI

The message brings about curiosity, it is difficult to be decoded. First of all we winder about what is written in there, then we understand that

$$\text{Ж} = (\& \times 3) – 1$$

represents the formula according to which the collection should be ordered. But what do those Ж e & mean?

In order to be able to manipulate them, pupils decide to substitute them with other symbols, graphically close to Brioshi's. They decide to write 'x' instead of Ж and 'z' instead of &:

$$x = (z \times 3) – 1$$

→

64. Texts in Japanese characters have been written with the help of a Nippologist friend; translations are reported straight after the text.

65. Translation:

Dear friends from Belluno, solve this problem:,

$$\text{Ж} = (\& \times 3) - 1$$

See you soon.
Brioshi

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
Many start writing equalities with the correct structure, substituting numbers for variables.

The following sentences are written on the blackboard:

Ж = (& × 3) – 1　　　x = (z × 3) – 1

8 = (3 × 3) – 1
5 = (2 × 3) – 1
2 = (1 × 3) - 1
17 = (6 × 3) - 1
11 = (4 × 3) - 1
20 = (7 × 3) - 1

Pupils were able to get to this point, but actually they made substitutions without being in control of their meaning. As a consequence they do not know how to place objects within the toys box because they do not know how to interpret writings on the blackboard.
Discussion evolves around equalities.
They understand that the numbers preceding the equal sign stand for Ж and they highlight them with a border. Then they realise that the first numbers inside parentheses stand for &, and highlight these too with a different colour *(here white and grey)*:

Ж = (& × 3) – 1　　　x = (z × 3) – 1

8 = (3 × 3) – 1
5 = (2 × 3) – 1
2 = (1 × 3) - 1
17 = (6 × 3) - 1
11 = (4 × 3) - 1
20 = (7 × 3) - 1

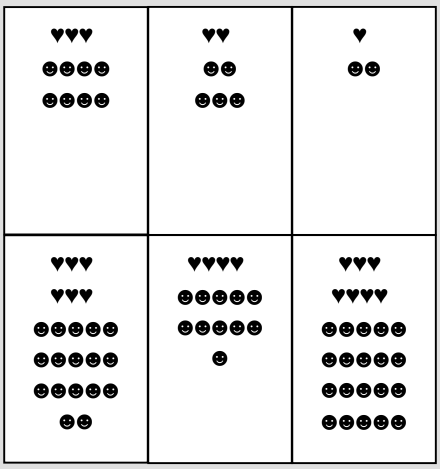
→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
At this stage pupils put marbles in the box, section by section, on the basis of numbers they find in the two highlighted sets . 66
The drawing is included in the answer to Brioshi and they also challenge him

Dear Brioshi,

We are your friends from the 5th grade in Belluno.
We understood your message and we solved it in this way, filling in 6 sections only:



Ora ti mandiamo il nostro:

$$(v \times 2) + g = 20$$

Vediamo come te la cavi!
La classe quinta B

66. *We report here an excerpt from the teacher's notes, illustrating clearly the dynamics that go together with the evolution of algebraic stuttering and the role played by the teacher herself around these dynamics:*

'Walking around the classroom I noticed that children at this stage were stuck. Also substituting Japanese letters for Italian letters children stopped to wonder: ".. what should I do now?" Some needed the little input: "try to think about letters as if they were numbers… try to put a number instead of "x" and "z" …" (this input might have been a big help for them). Many pupils, once overcome this obstacle carried on. Others substituted the letter for a random number, without respecting the law. Then they reflected on this point.
While helping a child I noticed an interesting thing: she started from "x" and carried on by trial and error. For instance, I try and put 8... good… because I do 3 × 3 - 1 o.k., I try and put 9... I can't … etc.
I was curious to ask pupils, some days later, how many of them started from "x" and how many from "z". I got the following answers:

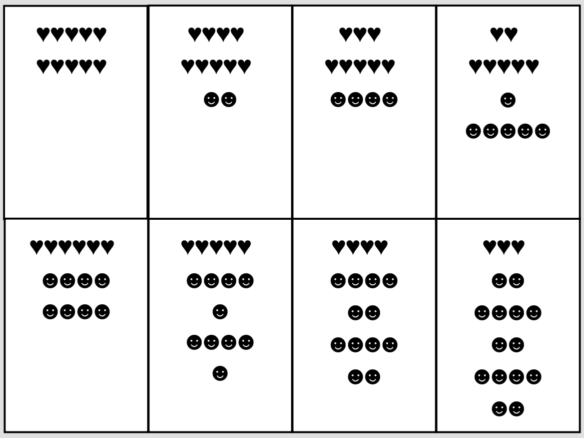
- *about 4 children starter from "x"*
- *2 children at first from "x" but they realised that it was easier from "z" and then they carried on from "z"*
- *the others from "z" because it is easier.'*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

Diary 29 (the same 5th grade as in Diary 28, January)

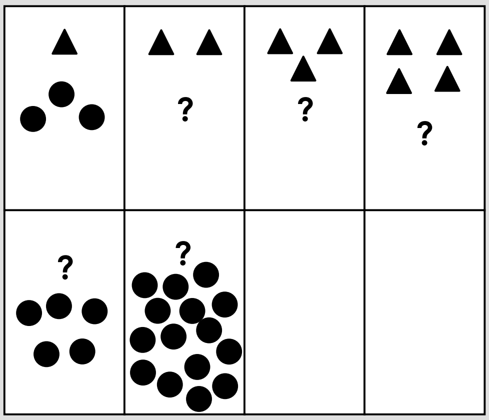
A week later Brioshi's reply arrives **67**:

ベッルーノの親愛なる友達へ
OK, 正解です！
君たちの問題が何なのか分かりました。



この問題に答えてください。

$$Ж = (\& \times 2) + 1$$



お元気で
君たちの友達より

BRIOSHI

67. *Translation:*
Dear friends from Belluno,
OK, it's right!
I solved your problem.

…

Now you solve this problem
$Ж = (\& \times 2) + 1$
Best wishes
Your friend
BRIOSHI

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

Usual curiosity and initial attempts to decode the message. First of all there is 'OK' in the middle of the sheet

«You understand that Brioshi is saying that our solution was right!»

We analyse his solution to our problem.

«Brioshi understood and he translated our letter '*v*' with the symbol ♥ and letter '*g*' with the symbol ☻»

We must check whether Brioshi answered correctly. We decide that in order to verify this, we must count objects within the sections, then insert numbers of objects in the equation and check the calculations. [68]

Again some pupils meet an initial difficulty, already met in the past: facing the equation $(v \times 2) + g = 20$ they do not double v, they calculate as if it were written $v + g$.

The problem sent by Brioshi is tackled, working in groups.

Pupils understand the problem and solve it correctly; generally they start from the <u>unknown</u> & and find the corresponding value of Ж.

Unexpectedly two pupils use the inverse process:

«I put an odd number instead of Ж, I take 1 out and I divide by 2. SO I find &, and Ж is always an odd number otherwise the operation doesn't work»

A pupil disagrees.

«If I find first & and multiply by 2 and then add 1 it's easier… you made a complicated tour!»

√ Pupils are asked how they solved the problem.

«We tried with the first example *(i.e. the first section)*, we tried to substitute Brioshi's letters for the number of drawings and we checked whether that was right»

Discussion highlights that many pupils have found a certain number of solutions and reported them in the toys box as functions of the 'visible' variable (triangles in sections 2-4 and circles in 5).

→

68. *In pratica è il cammino inverso a quello compiuto la volta precedente (descritto nel Diario 28), quando si erano usati i valori dei numeri dell'equazione di Brioshi corri-spondenti a Ж e a & per trovare il numero dei cuori e delle faccine dentro gli scomparti.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

→
The table of solutions for sections between 2 and 8 is written on the blackboard (for the last two, pupils invent a number for triangles).

	2	3	4	5	6	7	8
Ж	2	3	4	2	7	0	5
&	5	7	9	5	15	1	11

«Ж rappresenta il numero delle palline e & il numero dei triangoli»
«Le palline sono il doppio del numero dei triangoli più 1»
Si scopre che
«Non è uguale prima aggiungere 1 e poi fare il doppio o viceversa fare prima il doppio e poi aggiungere 1»
E poi un'altra scoperta:
«Il numero delle palline è sempre dispari»
«Sì, se hai un numero pari di palline non riesci a trovare un numero per i triangoli, se è dispari sì»
«Per forza, un numero moltiplicato per 2 è sempre pari e quando aggiungo 1 è dispari, perciò le palline sono sempre dispari» [69]
Un'ultima scoperta:
Nei comparti 2 e 5 ci sono gli stessi numeri, che però sono stati trovati in due modi diversi, attraverso la formula diretta e la sua inversa.

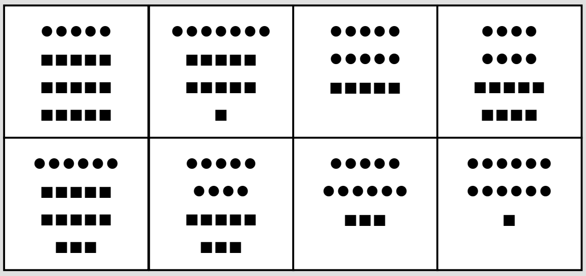
69. *Children find out that the image of the function <number of triangles , number of marbles > is the set of odd numbers and therefore the inverse correspondence becomes a function only if we operate on odd numbers (see **Note 3**).*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	Comments

Diary 30 (5th grade, February)

Another message from Brioshi **70**:

この問題に答えてください。



x + (2 × y) = ?

お元気で君たちの友達より

BRIOSHI

70. Translation:
Dear Friends from Belluno,
solve this problem:
x + (2 × y) = ?
Best wishes, your friend
Brioshi

We start decoding the message together.

«Squares are all odd numbers »

«In the first section there are 15 squares, that could be x, with 5 dots doubled we get 25, that is 15 + (2 × 5) = 25… *(chatting for a long time they make some trials)*. Always, if we take objects from each section and apply the rule, we always get 25»

The other pupils verify the rule too.

«It is not true that in each section there are 25 objects, but it is the rule that gives 25. »

«If x is the number of squares and *y* the number of dots, the result is always 25»

«*x* are squares, the number of squares, and *y* the number of dots, we counted the squares, which are 15, we added 2 times 5, because there are 5 dots, and only this is multiplied. We continued this way. All the results are always 25»

«If *y* is the number of squares and x the number of dots we get different results, in the first section we get 35 and then in other sections we get different results, here there is no rule, the actual solution to send to Brioshi is 25» **71**

«Let's write to Brioshi that the solution is 25»

71. This pupil enacts a strategy which can be similar to 'ab absurdum' reasoning in some ways. While his classmates repeat the same concept in different forms, he shows how, inverting attributions you always get different numbers, hence he 'proves' that the first hypothesis is correct.

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

15. Challenging messages

Brioshi, besides being a powerful educational mediator, can become an instrument for evaluating knowledge acquired by pupils. In this last case a situation of this type is illustrated: some classes who worked on the search for correspondence laws, are tested through a message exchange with Brioshi. This time there are no authentic messages but it is up to the teachers to evoke the Japanese friend as background for the activity. At this stage classes have understood the 'sense' of Brioshi, and they get immediately used to the proposed stimulus.
Problem contexts are analogous: some boys challenge their parents to find out the rules they have established to put their collections in order.

Diary 31 (5th grade, April)

√ 'Aurora's problem' is proposed:

Aurora challenger her parents to find out the rule on the basis of which she wants to classify her orange and yellow beads .
On a sheet left on the desk next to some empty boxes, she writes:

$$v = a + 3$$

Write down on a sheet, using natural language, the rule that parents must follow to fill in boxes with orange and green beads.

These are pupils' translations:

→

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
A) Correct, relational-type answers **[72]**
(they say 'what it is') (7 pupils)
- "the n of green beads equals the n. of orange beads plus 3"
- "Green beads are always 3 more than orange ones"

B) Correct, procedural –type answers
(they say 'how you do it') (5 pupils)
- "To find green beads you must do orange beads plus 3"
- "I add three to orange beads and I find green beads'
- "The n. of green beads can be found adding the n. of orange beads plus three"
- "If you make orange beads plus 3 you find green beads"
- "Aurora's rule is to make orange beads with other three and you find the green ones"

C) Containing the inverse relationship (1 pupil):
- "Orange beads are always 3 less than green ones"

D) Not understood situation (1 pupil):
- "A + 3 means 3 orange beads that have the value of 1 green one"

√ «How will Brioshi write the inverse rule?»
«a = v – 3!»

Diary 32 (4th grade, April)

√ «Try and translate in mathematical language the text written by Martina »

The number orange beads is 3 more than yellow ones

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72. *About this aspect see* ***Note 4*** *'Being' and 'doing'.*

Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

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At the end of individual activity a pupil raises her hand and goes to the blackboard to write:

The number orange beads is 3 more than yellow ones

a + 3 = g

√ «Good… let's think about it. From Martina's sentence do you understand that there are more orange or yellow beads?»
Pupils write down their answers and these are reported on the blackboard.

The number orange beads is 3 more than yellow ones

a + 3 = g

a > g *(9 pupils)* **a < g *(2 pupils)***

√ «Ok. The majority of you think that a is greater than g. But then is a + 3 = g correct?»
«From what Martina wrote, the number of orange beads, if I add 3, I get the number of yellow beads and then I translated a + 3 = g» **73**
«(Giulia) I don't think so… to me it is g + 3 = a»
The teacher adds the two writings at the blackboard…

The number orange beads is 3 more than yellow ones

a + 3 = g

a > g *(9 pupils)* **a < g *(2 pupils)***

g + 3 = a **a + 3 = g**

→

73. *It is a classic inversion mistake: translation in mathematical language is apparently the transcription of Martina's sentence.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
… and then she asks
√ «Are the two last sentences equivalent?»
The class is puzzled. There is an attempt:
«Yes, because there is equal…»
√ «What did Giulia write?»
«g + 3 = a»
√ «So: does this sentence translate what Martina says?»
The class reaches the conclusion that Giulia's sentence is true and that you must add 3 to yellow beads to get orange beads »

Diary 33 (5th grade, May)

√ The following situation is proposed:

In the kitchen, stuck on the fridge's door, Adriano left this message, as a challenge for his parents to find out the rule according to which he wants to classify his orange and yellow beads:

$$\boxed{a = 4v}$$

Translate the message in Italian language.

«But isn't there anything between 4 and v?»
√ «When in algebra you do not see anything between two letters there is always something. In arithmetic I must always write 3 × 4 = 12 otherwise it would become 34 = 12. In algebra you write 3 × a but mathematicians, to simplify, took × out and substituted it for a dot. Then they took also the dot out and wrote 3a. Now would you be able to write the rule?»
Translations classified by the teacher in the classroom are transcribed:

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Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

→
A) Correct, ‘ontological-type’
(they explain ‘what it is’) (7 pupils)
“The number of orange beads equals the green beads times 4”
“The n. of orange beads equals 4 times the n. of green beads”

B) Correct, ‘relational-type’
(they express the equality between two numbers) (1)
“To find the number of orange beads I must multiply 4 by the number of green beads”

C) Mixed (1):
“To find the n. of orange beads equals the n. of green beads times 4”

D) Not understood situation (2):
“The number of orange beads is the number of 4 green beads ”
“The number of orange beads equals green beads”

E) Not knowing what to write (3):
“The number of orange beads is the number of 4 green beads”
√ «If I have 2 green beads, or rather if I have 5, how any orange beads do I have?»

(The teacher records)

v	*a*
2	4 × 2
5	4 × 5

Diary 34 (5th grade, April)

√ The teacher is launching new challenges to the class «the other times I presented some situations in which something happened and we had to tell the facts searching for … »
«… the rule!»
√ «Now imagine that the blackboard is the computer screen through which we can communicate with Brioshi.

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Activities suitable for the classes	1	2	3	4	5	1	2	3	Comments

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We switch it on and you try to understand the message he sent to you, to interpret it and send the answer back to him, of course in mathematical language. Let's say that B stands for Brioshi and N stands for We» **74**.

B: a + b = 35 b = 18 a = ?

√ «Start to write the answer you could send to him. We will choose together the best fitting one. »
Proposals:

B: a + b = 35 b = 18 a = ?
N:
(1) a = 35 - b
(2) a + b = 35 b = 18 a = 35 - 18 = 17
(3) a + b = 35 b = 18 a = 270 [57]
(4) a + b = 35 b = 18 a = 17
(5) (a + b) - b = a
(6) a + b = 35 b = 18 - 35 a = 17

A discussion for the choice of the answer is enacted:
«(2) is the most correct but we did not need to repeat a + b = 35 because Brioshi knows that and the same for b = 18. If 17 is correct and it is a repetition, it would be like writing that 35 - 18 = 17»
«I wrote (3) and I made a mistake because I confused the 'plus' sign with the 'times' sign»
« (6) is wrong because we cannot do 18 - 35 »
«In (4) it was not necessary to repeat a + b = 35»
«In (4) you should put how you find 17»
« (5) has only letters and is correct »
→

***74**. The pupil wrote on his sheet in column the calculation giving as result 270: 18 × 15.*

Activities suitable for the classes	1	2	3	**4**	**5**	**1**	**2**	**3**	**Comments**

→
«Also (1) is correct because numbers and letters are used together »
√ The teacher deletes answers that can be taken out according to pupils and in the end the following two 'cleaned' writings remain:

B: $a + b = 35$ $b = 18$ $a = ?$
N:
(1) $a = 35 - b$
(2) $a + b = 35$ $b = 18$ $a = 35 - 18 = 17$
(5) $(a + b) - b = a$

√ The class' answer is thus written under Brioshi's message; to do this, we propose that all three sentences on the blackboard are used: (5) will work as 'general project', and from (1) onwards we will present the process followed to find the value of a.

B: $a + b = 35$ $b = 18$ $a = ?$

W: $(a + b) - b = a$
$35 - b = a$
$35 - 18 = a$
$17 = a$

A discussion is enacted about when letters are used instead of numbers.
«I use the letter when I must substitute a name, like black dots, and I write n»
«Otherwise when I do not have a number»
«When we are not able to know a number we use a letter so that we can find it»
√ «So, let's summarise: if we know a number it is useless to use the letter, right? We use it when we do not know the number. So in the message for Brioshi we could put 35 instead of a + b and 18 instead of b. Hence we use the letter to find… we have already called it like this …»
«… the unknown!»