AN EARLY ALGEBRA GLOSSARY AND ITS ROLE IN TEACHER EDUCATION[1]

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ABSTRACT
The paper deals with issues related to early algebra in the general framework of social and educational needs, as well as of competences indicated by the PISA test and the new role of the teacher. The main focus is on the theoretical framework of our ArAl Project – an integrated project for teaching research and curricular and methodological innovation in the classroom – through an analysis of its glossary's structure. An exploration of the linked universes of arithmetic and algebra and a method to carry out this exploration are proposed to teachers. The aim is to help teachers merge theory and practice, guiding them, through a reflection on their own knowledge, towards a conception of mathematics as a language, on the way of an approach to arithmetic thinking in an algebraic view. Finally, the paper proposes the use of the glossary in the teaching activity, as a cultural support for students as well as for teachers.

1. INTRODUCTION

Socio-constructive teaching, especially at compulsory school level, seems to be a most appropriate model to educate students to deal with the complex society we live in. Nowadays the labor market is highly competitive, continuously changing and is ruled by multifaceted systems of relationships: individuals are supposed to have diversified and flexible competences to both face and manage changes. Social educational needs are reflected in the international test PISA (OCSE 2002, 2003) which aims to test students at the end of compulsory school, regarding those skills which are viewed as essential for them to be active and conscious citizens in society. This test indirectly promotes a mathematical-scientific education that goes beyond the simple transmission of knowledge to be immediately applied, and rather aims at educating students to deal with complex situations. In the case of mathematics, wide space is given to problem solving, mathematization and interpretation of phenomena that involve different registers of representation (Duval, 2006). If we look at skills evaluated by the PISA test, some, like *argumentation, representation, communication* – intertwined with methodological aspects of socio-constructive teaching – are not generally acknowledged by teachers as specific to mathematics. Other skills, like *thought and reasoning, posing and solving problems,* are viewed as typically mathematical but 'high' and difficult to be reached by the medium-bottom set of students. The PISA test stresses *mastering of cognitive processes* more than understanding and application of concepts. Moreover, competences are assessed with reference to the level of complexity which causes their enactment. Particularly relevant and meaningful are those classified in the category of reflection that

[1] Due to the themes dealt with this contribution links to Malara and Malara & Navarra's contributions in this volume.

include creative thinking, intuition, generalization, enacting strategies and carrying out complex reasoning, besides reflection as usually viewed.

In this framework, the role of teachers becomes more complex and multifaceted and they become *decision makers* (Shulman, 1985: Cobb, 1988; Carpenter, 1988; Mason, 1994; Simon, 1995; and Malara & Zan, 2002). In everyday teaching, they face a number of different situations which force them to make decisions repeatedly. These decisions concern not only the schedule of classroom-based work and the identification of suitable tasks: they also refer to the choice of communicative strategies that should be used in class interactions (Bartolini Bussi 1998; Anghileri 2006), and involve both the identification of problems emerging from the class and their solution (Cooney & Krainer 1996, Jaworski 1998, Schoenfeld 1999).

This model of teachers requires competences that can be acquired with efforts and draw on various domains of knowledge: mathematical knowledge, often poor (Ball et al., 2001); general socio-psycho-pedagogical knowledge; mathematics education knowledge; specific contents, which can be referred to Shulman's pedagogical content knowledge (1986); knowledge of mathematics teaching and learning models.

Research has pointed to the strict relationship between quality of teachers' knowledge and quality of learning and highlighted how educational projects aimed at teachers, which make them access high-quality research literature, relevant to teaching and with an accessible language, produce deep changes in their conceptual views and strongly impact on their practice (Malara & Zan 2002 and related references).

These studies show optimal ways of organizing and developing teacher education and offer precious directions on how the latter should be implemented on a large scale.

In our country, this requires a deep change in the current formative and training courses, and dramatically stresses lack of solid institutional forms of in-service teachers' long term professional development.

One of the main problems teachers encounter in their professional life, in our country, is the progressive deterioration of their conceptions, as they get far from the years of their education. Real life – sometimes hard, often frustrating – gradually prevails over cultural and methodological references (assuming that teachers had the opportunity to have some, and this is not to be generally taken for granted). This deterioration has a number of reasons and each contributes to *impoverishment of identity*. In the case of mathematics teachers – but the same might probably be said for other disciplines – impoverishment is shown by a progressive disappointment about their own teaching instruments, regarded as incapable of having a positive influence on their students' construction of knowledge. Often results – either positive or negative – seem to emerge 'by nature,' *not depending on the teacher*.

Identity can be fulfilled only through the awareness of having some *points of reference*. In the case of teaching – due to its features of being linked to social changes and, at the same time, to the continuity or discontinuity of these transformations – these points of reference should be extremely *mobile*, in the sense that teachers should become sensitive enough to progress in these changes, always making a critical reflection on their own interpretation instruments.

Teachers often acquire these instruments – we might say: they construct their own epistemology – during their studies, from infant school to graduation or post-graduate certificate. After this, they immerse in real life, slowly loosing contact with their theoretical studies, and regard the latter as a foundation set *once and for all*, frozen in stereotypes and impoverished by lack of renewal.

Negative results of tests like PISA dramatically stress the outcome of these attitudes, and initiatives, like the PDTR project, attempt to build up strategies to fight this, joining the efforts of participant countries.

This paper was devised as an answer to these issues and remains within the framework of the studies carried out by our group for the renewal and re-qualification of teaching of arithmetic in a pre-algebraic perspective and for an anticipated approach to algebra viewed as a tool for thinking. It aims to provide a contribution to mathematics teachers – almost a challenge – towards *enrichment in their identity*. *An exploration*, or better a *re-examination*, of wide conceptual universes, such as the arithmetic and algebraic ones, and a *method* to carry out the exploration will be proposed. We will use the tools elaborated within the ArAl project (Malara & Navarra, 2003; Navarra & Giacomin, 2004-2006; Malara et al., 2004, Fiorini et al., 2006).

2. THE ARAL PROJECT AND EARLY ALGEBRA

ArAl is a project aiming at *didactical innovation* in mathematics, designed for students aged 5-14, and framed within the early algebra theoretical framework. Basic aim of the project is to design and implement teaching sequences in arithmetic, in a *view that favors an anticipated approach to algebraic thinking*, aiming at a progressive construction *in parallel* with arithmetic and *not successively* to it. These sequences are meant to contribute to a reduction of difficulties – when they first appear in the arithmetic field – many 15-year-old students encounter in the study of algebra when they enter secondary school and that often turn into absolute obstacles. At the same time, these sequences aim to justify to students the role played by algebraic language *in modeling problem situations* as well as in the *production of thinking*.

Our hypothesis is that, starting from early years in primary school, the teaching of arithmetic should be implemented with the aim of letting students learn mathematics as a new *language* with modalities that resemble those used in the learning of a natural language. For this reason, we introduced the concept of *algebraic babbling*. Socio-constructive educational modalities underpin the development of this hypothesis: teachers do not transmit knowledge to be learned but rather devolve to students the construction of knowledge, during a *collective* interaction that starts from the exploration of appropriate problem situations.

3. THE TEACHER

Mathematics teachers are *the key element* to the enactment of these changes of perspective. The challenge is to lead them to *revise* the roles they *are already playing*, towards a critical reading of knowledge, beliefs and maybe stereotypes. Therefore, participating in the ArAl project is for teachers an important moment for reflecting upon central issues, like: *Which* arithmetic am I teaching? *Which* algebra? *When does 'algebra' start?* These questions concern both primary and lower secondary teachers, who, in most cases, do not have a background in mathematics (neither regarding contents, nor the teaching/educational aspects), and higher secondary teachers, often lacking in terms of teaching/educational aspects.

In the theoretical framework and instruments of the ArAl project, teachers find the necessary support for revising arithmetic and algebra, enacting, in some respect, an actual "Copernican revolution" in their mathematical culture. They actually have to come to deal with a wide, complex and somewhat maze-like set of concepts, which confuses them and makes it difficult to embed the early algebra perspective in their everyday teaching. The complex nature of this situation requires a conceptual support that may enable teachers to cast light upon their formative route. A glossary provides this support, helping teachers merge theory and practice and in particular; face the connections between mathematics and linguistics that characterize the ArAl project's theoretical framework; which leads them to a *conception of mathematics as a language*, in which a convincing *control of meanings* can be developed and transmitted to students.

4. EXPLORING EARLY ALGEBRA AS A MACHINE

Early algebra is a polycentric universe of themes. This can confuse teachers and make their approach to these themes difficult. It is therefore necessary to elaborate a framework to help teachers gradually achieve an organized vision of early algebra, through a reflection upon their own knowledge. Pre-defined approaches do not exist. There are a plurality of possible approaches that require reflection and method.

We thus analyze early algebra through the metaphor of a "machine," the functioning of which we want to find out. Recomposing the machine's devices is an individual adventure, and depends on how teachers decide to explore it. These relate to school level, features of the particular class, teachers' expertise, studies, attitudes and curiosity that lead them to approach some particular themes before others. Therefore, we are talking about a method that each teacher might apply, depending on both their individual and environment-related needs.

5. THE GLOSSARY

The starting point for observing our machine's devices is therefore in the glossary, currently made of 92 key terms. The description of each key term leads to other key terms. For example, the term *argumentation* leads to: 'collective, process/product, representation, relationship, semantics/syntax' (Figure 1):

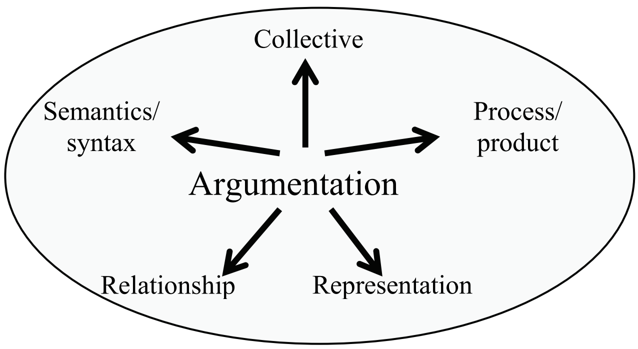
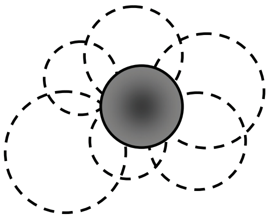


Figure 1.

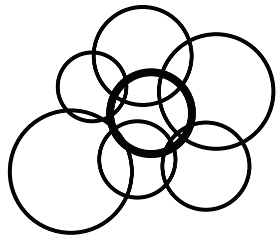
Let us call this representation a *net* of the term 'argumentation.'

Assuming the glossary as *matrix of nets*, we define a net as "the complex of references that connect a term to other terms of the glossary." Every net can be considered in two ways:

(a) through its *inner* structure
(a universe in itself)



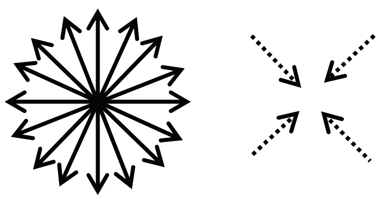
(b) how it links to *other* nets.



These considerations lead us to reflect on *the structure* of the glossary. Every term is characterized by two components: (1) the numerousness of its net; and (2) the numerousness of its occurrences.

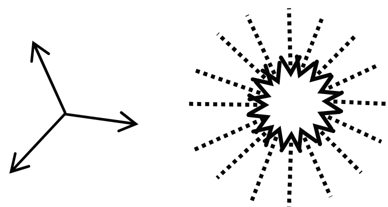
Two examples:

The definition of the term ‘didactical mediator’ contains a high number of glossary-terms (15) but only 4 occurrences in other nets:



The wealth of “didactical mediator”
is due
to the numerousness of its net

On the contrary, the net of ‘mathematical phrase’ is poor in glossary-terms (only 3) but it appears in 17 other nets, and therefore its significance is of great transversal importance:



The wealth of “mathematical phrase”
is due
to the numerousness of its occurrences.

This first classification allows us to understand, for instance, that ‘didactical mediator’ works better than ‘mathematical phrase’ as a possible starting point for an initial approach to early algebra because, due to the numerousness of its net, it locates teachers within a double process of conceptual *deepening* and *extension.* Deepening of the term occurs through an illustration of the relationships that link the 15 key-terms in its definition; extension occurs because each one of the 15 key-terms is a potential stimulus to read the related definition. This definition, in turn, contains a new net, a possible basis for further exploration. Starting from ‘didactical mediator,’ teachers can thus construct their *own personal organization of the early algebra knowledge*, going through successive choices of key-terms and moving from definition to definition. At the same time, ‘didactical mediator’ is not often quoted in other nets and it is thus difficult to be found. We might say it is a mountain refuge “reached by few trails, but starting point for many excursions.”

‘Mathematical phrase’ on the contrary, leads to few nets (three only), but can be found in 17 of them. Therefore it suggests, through the second word, the *transversal* importance of the *linguistic* aspect in the conceptual structure of ArAl Project. It is a refuge “where many trails converge and from which few, but potentially panoramic, footpaths leave.”

In order to move around in this complex universe, it is appropriate to identify a *reading key*.

6. A READING KEY TO THE GLOSSARY

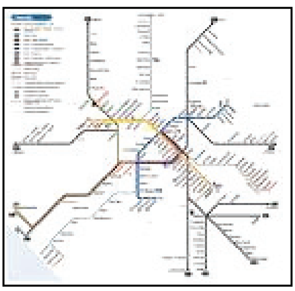
Our view is to identify a reading key with which teachers might get to know whatever they will be able to do in that particular moment of their journey, and through which they might learn to move around within the *local/global* pair along two directions: (1) inside a single *local* (a key-word); and (2) in the map of the possible connections among the locals which, rather than being approached through predefined ways, can be approached through a method that allows them to be explored.

As we said earlier, the exploration leading to a re-composition of the machine’s devices is an individual adventure and depends on how teachers *autonomously* decide to interact with the themes of early algebra. The objective is for them to gradually get to an increasingly organized and complex global view of the universe they approach. This should happen through pathways that might be heterogeneous and fractioned in time, as well as through a reflection which might also provoke a rupture with the teacher’s pre-existing knowledge and beliefs.

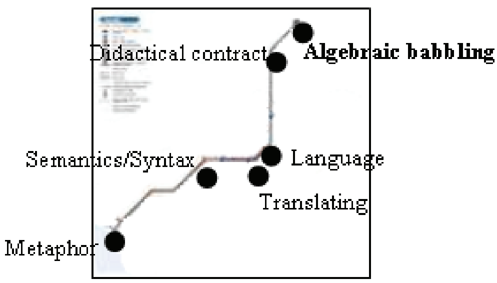
Our proposal links back to ancient problems related to the organization of knowledge, which mainly emerge when the studied topics and disciplines are wide and with hazy boundaries. These problems are linked to the need of searching for mutual influences, similarities and oppositions, inner logic of spheres of knowledge which often went through a different historical development and refer to different epistemological statutes. Applying this to the case of early algebra in the Aral Project’s perspective means exploring the *connections between mathematics and linguistics* that may favor a conception of *mathematics as a language* in which a convincing control of meanings can be developed by teachers and, later, by students.

Our main objective is, therefore, not to provide models of behavior, but rather to lead teachers to a concrete ground on which they might reflect on the “grammar” needed to move within the conception of early algebra. In this context the glossary represents a *textual map* aiming to:

(i) favor a *global* vision of the teaching activities’ theoretical framework:



(ii) define *local* situations made of pathways (the nets) linking single terms to one another:



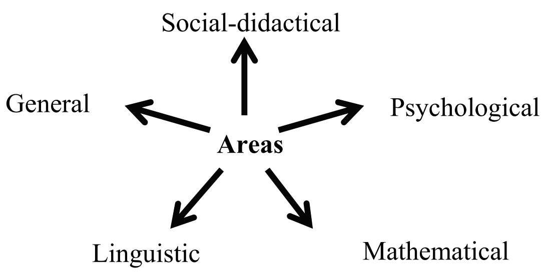
Each of the several possible routes of the glossary allows for a different reconstruction of the *sense* of the theoretical framework and of the links between reference areas (general, social, psychological, linguistic and mathematical) we will analyze in the next paragraph.

We remark that the main objective is to favor teachers' search for a necessary synthesis between theory and teaching practice. In the case of the route in the example, the synthesis of the net's terms may be:

> 'Algebraic babbling' is a 'metaphor' which puts side by side learning modalities for 'natural language' and those for 'algebraic language.' Through a suitable 'didactical contract,' which tolerates 'syntactically' promiscuous initial moments, it favors 'translations' between the two languages.'

7. AREAS IN THE GLOSSARY

Key-terms fit into five areas:



Some examples from the five areas are:

> *General area*: 'Brioshi, didactical mediator, opaque/transparent (with respect to meaning), process/product, relational thought, representing/solving.'
> *Linguistic area*: 'algebraic babbling, argumentation, canonical/non canonical form, language, letter, metaphor, paraphrasing, semantics/syntax, translating.'
> *Mathematical area*: 'additive/multiplicative form, equals sign, formal coding, mathematical phrase, pseudo equation, regularity, relation, structure, unknown.'
> *Social-didactical area:* 'collective (exchange of views, etc.), didactical contract, discussion, negotiating, sharing, social mediation.'
> *Psychological area*: 'perception, affective/emotional interference.'

General, psychological and social-didactical areas provide a *methodological support* to linguistic issues, which, in turn, represent a *conceptual junction* towards understanding of the mathematical area (Figure 2):

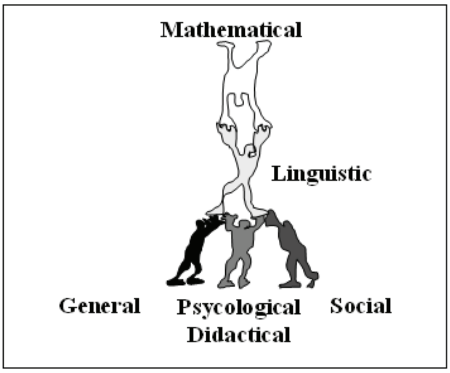


Figure 2.

In fact, since ArAl is a project concerning mathematics education, it is certainly true that the four 'bearing' components have a fundamental importance.

An early approach to arithmetic in an algebraic view is founded on a solid basis made of social and psychological assumptions as well as of a series of general basic concepts, which, in turn, sustain a strong *linguistic* component.

Teachers have to learn to promote and manage these supports, becoming aware that: (i) knowledge can be constructed through promotion of social processes that favor both *exchanges of ideas* and *verbalization* in the classroom; (ii) identifying suitable *didactical mediators* (for instance, the virtual Japanese student Brioshi who, not knowing languages other than his, 'forces' Italian students to use a correct mathematical language when they exchange messages with him) is essential to a stable acquisition of *meanings*; (iii) it is necessary to promote activities that enhance *metacognitive* and *metalinguistic* aspects.

Becoming aware of this is a fundamental prerequisite for teachers to organize and manage more specific mathematical activities.

Should these basic aspects be overlooked or developed only superficially and in a fragmentary way, the project's cultural character would be impoverished and potential didactical results invalidated.

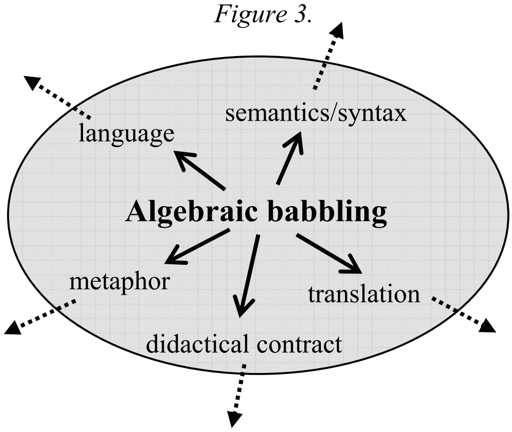
8. THE MATRIX OF NETS AND OCCURRENCES

Some terms in the glossary are particularly important. To illustrate this point we make use of a matrix (see next page) which, due to space reasons, only refers to a selected part of the glossary terms (about one third).[2] Nets of terms can be read in the rows, whereas columns contain occurrences, i.e. how many times terms are quoted in the nets. A net is thus a space that can be analyzed fully as well as in some particular areas.

Let us compare three nets: that of the 'algebraic babbling' construct, and the pair made of the nets of 'argumentation' and 'collective.'

[2] Inconveniences caused by this selection are that some nets are more numerous than they appear in the matrix. Numbers in the first row and in the last column indicate the real numerousness of the respective occurrences and nets, but in some cases the number of correspondent crosses is smaller, because many connections are excluded from the matrix. We believe that despite this limitation the matrix is still meaningful as a support to what is discussed in the text.

Figure 3.



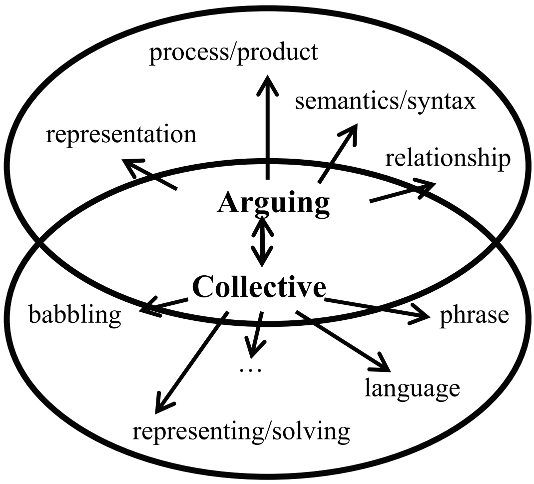
As concerns 'algebraic babbling' (Fig. 3), the five terms in its net ('semantics/syntax, translation, didactical contract, metaphor, language') lead to as many nets, thus widening its conceptual horizon. In the matrix, the five connections are highlighted in the row of 'algebraic babbling' with lightly marked crosses.

MATRIX OF NETS AND OCCURRENCES OF SOME TERMS IN THE ArAl GLOSSARY	Additive / multiplicative	Argumentation, argumentation	Algebraic babbling	Brioshi	Canonical / non canonical	Formal coding	Collective (discussion)	Didactical contract	Thrill from symbols	Mathematical phrase	Unknown	**Letter**	Language	Didactical mediator	Metaphor	Negotiation	Opaque/transparent	Paraphrase	Principle of economy	Procedure	Procedural	Process/product	Protocol	Pseudo equation	Representing/solving	representation	Relational (thinking)	Relationship	Semantics/syntax	Structure, structural	Translating/translation	Equals (sign)	Verbalizing, verbalization	Numerousness of nets
Numerousness of occurrences	**16**	**3**	**4**	**9**	**5**	**2**	**8**	**4**	**3**	**17**	**3**	**10**	**20**	**4**	**3**	**1**	**5**	**2**	**1**	**6**	**4**	**12**	**4**	**1**	**14**	**8**	**5**	**10**	**11**	**5**	**12**	**2**	**1**	
Additive / multiplicative (form)	■									X											X	X				X			X					**5**
Argumentation, argumentation		■					⊠															X				X		X	X					**5**
Algebraic babbling			■					X					X		X														X		X			**5**
Brioshi	X			■	X				X			⊠	⊠	⊠											⊠	X					⊠			**8**
Canonical / non canonical (form)					■												⊠					X			X	X								**6**
Formal coding						■				X	X	X	X					X								X		X			X			**8**
Collective (discussion)		⊠	X				■			X			⊠										X		X				X		X		X	**10**
Didactical contract								■																										**0**
Thrill from symbols			X		X		X	X	■	X		X								X					X				X					**10**
Mathematical phrase										■			X													X			⊠					**3**
Unknown			X	X			X		X	X	■	X	X	X			X			X			X		X	X			X					**15**
Letter			X	⊠			X	X		X		■	X	⊠		X								X		X			X					**12**
Language				⊠			⊠						■	X														⊠						**4**
Didactical mediator	X			⊠	X				X			⊠		■	X		X									X	X		X					**15**
Metaphor															■		X					X												**2**
Negotiation		X					X									■										X								**3**
Opaque / transparent (meaning)				X	⊠					X			X				■					⊠			X	X								**7**
Paraphrase				X						X			X					■								X		X		X	X			**8**
Principle of economy												X							■			X			X					⊠				**4**
Procedure						X				X			X							■											X			**4**
Procedural																				X	■						⊠							**2**
Process / product										X			X				⊠					■			⊠			X		X	X			**7**
Protocol		X						X												X		X	■											**4**
Pseudo equation							X																	■	X	X					X			**5**
Representing / solving				⊠						X			X								X	⊠			■		⊠	X			X			**8**

Representation																										■		**X**						1
Relational (thinking)	X				X					X	X	X	X								**X**				**X**	X	■	X				**X**		11
Relationship	X												**X**													**X**		■						6
Semantics / syntax										**X**			X																■		**X**			3
Structure, structural																			**X**		X						X	X		■				4
Translating				**X**									X													X			**X**		■			4
Equals (sign)										X															X		**X**					■		3
Verbalization						X				X			X							X		X									X		■	6

The situation for terms ‘argumentation’ and ‘collective’ is more complex: in the net of ‘argumentation’ we find the term ‘collective’ and in that of ‘collective’ we read the term ‘argumentation.’ We might say that there is a sort of one-to-one correspondence between these two terms. Cross-references in both directions, as in this case, are particularly interesting: they stimulate a process of in-depth analysis/extension which might become a fruitful instrument to deal with phenomena of circulation around a certain topic. In the matrix, pairs of terms that have this special feature are those which rows and columns meet in cells with a cross marked in bold.

Figure 4.



The exchange of cross-references is the *strongest possible connection between two nets*. In the case of the pair ‘argumentation-collective’ the importance of communication, and consequently of language (‘argumentation’) in the social construction of knowledge (‘collective event’) is once again emphasized. Examples of crossing nets often refer to Brioshi (‘Brioshi-Letter,’ ‘Brioshi-Language,’ ‘Brioshi-Translating’), and this is easily understandable, since the virtual student’s role is that of a linguistic mediator. Other pairs show a less clear connection, and this makes the search more interesting: for example, in ‘canonical/non canonical form-opaque/transparent’ pair the connection becomes clear when we find reference to the *dual* term ‘process/product’ and the consequent associations ‘non canonical form – process – transparent’ and ‘canonical form – product – opaque.’

Concluding, some terms seem to be *denser* with relationships than others. We find here two alpine refuges “each being a starting basis for goals that include the other refuge among the possible excursions.”

9. KEY-TERMS IN THE GLOSSARY

The matrix, besides providing interesting elements for a general outlook on the world designed by the glossary, also suggests another investigation.

Let us select the terms having nets and occurrences equal or greater than 10 (related numbers are highlighted in dark cells, in the first row and in the last column on the right side). Then let us put these terms into a table, grouping them on the basis of the area they belong to (see next page; colors of cells, from black to white are the same as those used for men in Fig. 2).

There are 16 terms, all leading to interesting readings.

Issues	**Nets with more than 10 cross-references**	**Terms with more than 10 occurrences**
General	Mediator 'Thrill from symbols'	Process/Product Representing/Solving Representation
Social/didactical	Collective	
Linguistic	**Letter**	**Letter** Language Semantics/Syntax Mathematical Phrase Translating
Mathematical	relational unknown	Additive-multiplicative Relationship

First remark: these 16 – either terms or pair of terms – are the only ones that appear in *all* the nine ArAl Units also published in English. This makes them highly representative of the project's theoretical framework.

There are 6 nets with more than 10 cross-references, as shown by the matrix's last column. They play a key role in the project's theory. We illustrate the latter by underlining the terms included in the table.

Nets column: In an environment characterized by a strong intertwining of mathematics and language, identifying mediators as bridges between the two disciplines is a fundamentally important *general* need. This is also the case for social aspects concerning teaching educational processes enacted with the aim of achieving a collective – and thus shared – construction of knowledge. The term 'letter' might be seen as *expected* (we will get back to this later) and inevitable difficulties in its use emerge ('thrill from symbols, unknown'). The sixth term, 'relational,' is also a paradigm of the project's main aim, that is, to favor the development of relational thinking, thus overcoming a view of the *local*, as well as a search for outlooks on *modeling* and *generalization*.

Moving to occurrences, we notice that 'letter' appears again: it is the *only* term with nets and number of occurrences greater than 10. Together with its paraphrases, (indicator, initial, place card), it is the main 'road junction' in the glossary. The peculiarities of 'letter' we are reconstructing, confirm that we are at the boundaries between language and mathematics and strongly point out the ArAl project's theoretical assumptions. We are led to pose three basic questions: (1) What does the letter *represent* when it is used in mathematics? (2) How can a “model of teaching letters” be developed? Can the letter as a mathematical object be intuited and its meaning be acquired through exercise, as it happens in traditional teaching, or rather is it necessary to use *mediators* that favor a gradual understanding through slow successive steps, as maintained by the ArAl project? (3) A correct use of letters, in particular within a formalized language, is based on a system of *rules*. In order to understand their importance and necessity is exercise enough, or rather are other strategies needed? As explained earlier, in the ArAl project, the mediator Brioshi is used to this purpose.

Therefore ArAl’s point of view is: the approach to letters is neither simple nor intuitive, their meanings become clearer and more solid through exercise, but only on the surface. They must necessarily be constructed through suitable mediators that favor students’ gradual awareness of the system of rules underlying the construction of representations in a mathematical language.

‘Letter’ is the main junction for ‘conceptual sorting.’ The table shows how the relationship/relational area and three pairs: ‘process/product,’ ‘representing/solving,’ ‘semantics/syntax,’ are junctions for *conceptual exchanges*.

Unknown and didactical mediator are the terms with the most numerous nets (15); language’ is the one with most occurrences (20); Brioshi (net with 8 terms, 9 occurrences) confirms its central role.

The construction of a deep understanding of early algebra and of its implications for practice mainly focuses on control of these terms.

The following is an attempt to use the terms in the table to compose a synthetic definition that gathers them in a sort of temporary manifesto:

Foreword

The early algebra theoretical framework supports the hypothesis that students’ weak control of the meanings of algebra, originates from how arithmetic knowledge is constructed, starting from primary school.

Algebra should be taught as a new ‘language’ one appropriates– through a number of shared ‘social’ practices (‘collective discussion, verbalization, argumentation’) – with modalities that resemble those of natural language learning: one starts from meanings (‘semantic aspects’) and gradually locates them into their ‘syntactic’ structure (‘algebraic babbling’).

Fundamental elements to this purpose are ‘metaphors,’ didactical ‘mediators’ to the acquisition of meanings during the conceptual progression towards generalization and modeling.

In this view, a natural language is the most important mediator in students’ experience and their main instrument of ‘representation,’ enabling them to illustrate the system of ‘relationships’ (initially ‘additive’ and ‘multiplicative’ ones) linking elements of a problem situation. This causes a shift of attention from ‘product’ to ‘process,’ and induces a ‘translation’ into a ‘mathematical phrase.’ In this way, attention is shifted from the *arithmetical* objective of ‘solving’ to the *algebraic* one of ‘representing.’ At the same time, mediators favor the acquisition of the use of ‘letters,’ initially viewed in their most approachable meaning of ‘unknown.’

10. THE GLOSSARY AND TRAINEE TEACHERS

The glossary provides a support on which teachers can plan and manage the didactical transposition of the subject’s contents, particularly those concerning arithmetic and algebra. The glossary is a leading thread for teachers to acquire this skill: this occurs throughout all phases of the training program for teachers – either in-service ones like those involved in the PDTR project, or those attending the specialization courses in mathematics education held at Modena University by Malara and her staff (Iaderosa, Gherpelli, Nasi, and Navarra): theoretical lectures, laboratory-based activities and training activities carried out in the mentors' classes. Teachers are supposed to complete specific assignments aimed at testing their degree of awareness in relating the management of teaching activities to the reference theoretical framework.

We report here an example of a concluding reflection, written by a trainee teacher for an exam. The text deals with a classroom-based episode, taken from one of the many experimental ArAl teaching activities, audio recorded and proposed in *six enchained scenes*, sequentially presented with twenty minutes intervals in between. The trainee teacher is supposed to analyze each scene and complete the related tasks; after the first scene's analysis he moves to the second one and so forth up to the last scene.

> The structure of this didactical pathway seems meaningful as it stresses:
> 1. 'socially shared construction of knowledge,' enacted through a constant request of 'verbalization' and 'argumentation' by students and a 'didactical contract' characterized by the task "first represent, then solve;"
> 2. use of the symbolic representation register and of its supporting laws (the importance of 'language' and of the process of 'translation' from natural language to symbolic language and vice versa are particularly stressed);
> 3. use and analysis of 'protocols' and diaries, which enable teachers to draw information about their students' mental processes (internal representations) and, consequently, about their knowledge construction. While verbalization and argumentation help students acquire higher 'metacognitive' and 'metalinguistic' knowledge, protocols' analysis help teachers have a clear idea of the process through which students have constructed their knowledge and therefore make suitable didactical changes, when necessary;
> 4. risks of 'semantic persistence' in the use of letters.
>
> As concerns representation methods, the structure of the assignment:
>
> 5. enables teachers to deal with both 'opacification' and 'transparency' of mathematical phrases and their aim ('relational or procedural');
> 6. puts the accent on the search for 'relationships' as well as on their 'representations.'

11. CONCLUDING HYPOTHESIS: THE GLOSSARY AND STUDENTS

The glossary, as we said earlier, was mainly envisaged for teachers. It was a final sentence in the assignment of a trainee teacher during one of our postgraduate courses that struck us and led us to formulate some hypotheses on a possible widening of the glossary function as an *instrument of cultural support to students*. The sentence was:

> "I think that, for a teacher, experiencing is the most formative aspect*, especially if each one tries to 'show' his/her teaching style continuously*."

"Showing one's teaching style continuously." Possibly beyond the author's intentions, there is a strong implication of this statement: mathematics might be taught, making students aware of how the *fundamental basis of a meaningful* construction of mathematical knowledge is actually drawing on aspects that are apparently *outside* mathematics itself. Some of these aspects are: competence on the use of languages, starting from a natural language; being able to translate from one language to another; the importance of syntactic and semantic aspects of a language; the difference between representing and solving a problem situation; learning how to distinguish process and product.

This view led us to formulate a hypothesis on the widening of the glossary's function: we might think of a didactical contract that requires a *constant explicit statement of the deep motivations that guide teacher in their methodological and* *content-related choices*. In this way, students would view themselves as sharing the construction of knowledge and the glossary would become *a permanent background to teaching and learning*. Through the glossary students would be led to reflect upon the importance of sharing – with both peers and teachers – the *sense* of key terms, such as 'canonical form /non canonical form, letter, metalinguistic-metacognitive, opaque/transparent, principle of economy, argumentation, etc.'

An inevitable premise to this is that teachers come to be the prime movers of this sharing process, and therefore become conscious and convinced actors in the management of the glossary.

The concluding slogan might be: *to educate metacognitive students, we need to form metacognitive teachers.*

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