ASPECTS OF GENERALIZATION IN EARLY ALGEBRA

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In this paper we will present some studies we have recently developed in our research project in the theoretical framework of early algebra. We will illustrate a first "inventory" of those conditions which might foster the construction of significant basis to support young students' gradual approach to generalization from different points of view (linguistic, perceptive, social and mathematical).

INTRODUCTION: GENERALIZATION AND EARLY ALGEBRA

Traditionally, most curricula separate the study of arithmetic, mainly taught in primary school, from the study of algebra, considered to be suitable for secondary school students. However, many researches have shown the negative effects of a too quick transition from arithmetic to symbolic manipulation. Warren et al. (2006), for example, suggest that algebraic activity can occur at an earlier age and that this kind of experiences, proposed through appropriate teacher actions, could assist students in this complex transition. Blanton and Kaput (2011), too, stressed the importance of giving children opportunities to begin using symbolic representations as early as first grade in order to make them acquire those basic concepts which can allow them easily explore more complex concepts in later grades. These ideas have brought to the rise of early algebra, which now has the characteristics of a real new discipline (Kaput et Al. 2007). This is the frame in which we have developed our *ArAl Project* (Malara & Navarra, 2003) [1].

The hypothesis of early algebra is that the common "arithmetic to algebra" framework is too limiting and narrow (Smith & Thompson, 2007) and that therefore it should be reformulated in order to give students the opportunity to develop algebraic thought when they start carrying out the first activities in arithmetic. This approach does not require to bring the algebraic curriculum in primary school, but to revise the way in which arithmetic is conceived and taught in order to promote a shift from a procedural conception of arithmetic to a relational and structural one. We believe that it is also necessary to clarify what is the meaning of promoting the development of algebraic thinking at this level. We agree with Radford (2011), according to whom the use of notations is neither a necessary nor a sufficient condition for thinking algebraically and that algebraic thinking is characterised by the specific manner in which it attends to the objects of discourse. The author suggests that algebraic thinking is about

[1] ArAl Project (Arithmetic pathways towards favouring pre-algebraic thinking) is a National Project developed by the GREM (Group for Research in Mathematics Education) directed by N. A. Malara (professor in the Mathematics Department of Modena and Reggio Emilia University) and coordinated by G. Navarra.

dealing with indeterminate quantities conceived of in analytic ways (i.e. considering the indeterminate quantities as if they were known and carry out calculations with them as with known numbers).

Fostering the teaching of early algebra means, for teachers, giving their students the opportunity to activate different modes of thinking such as: analyzing relationships between quantities, predicting, generalizing, exploring stimulating situations, modelling, justifying, proving.

Generalization is considered to be an important determiner of growth in algebraic thinking and a fundamental preparation for later learning of algebra (Cooper and Warren, 2011). A rich context from the point of view of the different meanings that could be conveyed through it, and therefore potentially suitable to stimulating generalization processes, is represented by activities related to the research of regularities (see paragraph D2). During this kind of activities students have the possibility to experiment a crucial aspect in the generalization processes: seeing a generality through the particular and seeing the particular in the general (Mason, 1996). Cooper and Warren (2011) suggest that, during these activities, a step towards full generalization in natural language and algebraic notation is *quasi-generalization*, in where students are able to express the generalisation in terms of specific numbers and can apply a generalisation to many numbers, and even to an example of 'any number'.

In the approach to early algebra teachers play a crucial role in identifying the best activities to be performed and in promoting those processes which foster generalization. Obviously their way of proposing these activities in their classes is strictly connected to their deep beliefs, which have been highlighted thanks to our analysis of the numerous transcripts (about 4500, collected from 2004 and 2011) of the activities performed in our project. These transcripts were object of a joint reflection carried out by teachers and researchers through the *Multicommented Transcripts Methodology (MTM)*[2]. Some reflections on methodological aspects recur independently of the age of students (from 5 to 15); therefore they can be considered mirrors of the most widespread behaviours of teachers. The high number of reflections referred to generalization from different points of view suggested us to identify a tentative but enough detailed "inventory" that we will present in the second part of the paper. Before proposing this inventory it is necessary to introduce some theoretical aspects which constitute our framework for the approach to early algebra, with particular reference to the aspects related to generalization processes.

[2] The MTM, developed in the ArAl Project, is based on the critical analysis of transcripts of the audio-recordings of whole-class discussions, carried out by the teachers involved in the Project, through the intervention of different actors: the class teacher, his/her E-tutor, other teachers, teachers-researchers and university researchers. The commented transcripts are shared through E-mail and during periodical meetings for a critical exchange.

OUR APPROACH TO EARLY ALGEBRA

Our perspective in the approach to early algebra is *a linguistic and metacognitive one* and is based on the hypothesis that there is a strong analogy between modalities of learning natural language and algebraic language (Cusi, Malara and Navarra 2011). In order to explain this point of view, we make use of the metaphor of *algebraic babbling*.

This metaphor represents the process through which the student acquires first a semantic, then a syntactic control of the mathematical language in a way similar to the one he/she learns natural language. This learning is first characterized by an initial discovery of meanings and a gradual, creative appropriation of rules and by a subsequent deeper knowledge, developed during the school years, when the student is able to reflect upon the structure of the language.

Fostering this process requires to build up an environment able to stimulate the autonomous elaboration of formal codings, to be negotiated through class discussions, and a gradual experimental appropriation of algebra as a new language. The rules of this language are then located into a didactical contract, which tolerates initial moments of syntactic 'promiscuousness'.

Another fundamental aspect in our approach to early algebra is therefore recognizing the potential role played by the relationship between argumentation and generalization in the social construction of knowledge. Only when argumentation becomes a shared cultural instrument in the class this relationship can be made explicit and the students can understand the role played by verbalization in the development of their capability of reflecting upon what they are saying. Moreover, comparing particular cases help students recognize their similarities, gradually highlighting their connecting thread.

An other crucial aspect in this approach to early algebra is helping students recognize and interpret *canonical and non canonical representations* of numbers[3] in order to make them build up the semantic basis for the understanding of algebraic expressions. Non canonical representations can be considered "semantical ferries" towards generalization (see paragraph A2).

Because of the central role played by verbalization in supporting the achievement of symbolic notation, an other critical aspect is making students understand the importance of respecting the rules of algebraic language.

While students start soon interiorizing the importance of respecting the natural language's rules in order to facilitate communication, it is difficult to make them develop a similar awareness in relation to algebraic language. It is therefore necessary to help them understand that algebraic language, too, is a finite set of arbitrary symbols which can be combined according to specific rules to be respected. This kind of conception could be fostered through the creation of

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[3] Among the possible representations of a number, one (for instance 12) is its name, called canonical form, all the others (3×4, (2+2)×3, 36/3, 10+2, …) are its non canonical forms, and each of them will make sense in relation to the context and the underlying process.

linguistic mediators which force the respect of rules in communicating even advanced concepts by means of algebraic language, in a perspective which foster generalization[4].

FACTORS WHICH CONTRIBUTE IN STUDENTS' CONSTRUCTION OF THE SEMANTICAL BASIS FOR GENERALIZATION

As researchers who develop their studies in the field of early algebra, having to face the theme of this conceference (*Generalization in mathematics at all educational levels*) made us try to identify what kind of situations, methodologies and attitudes could foster, in young students, the construction of the significant premises for a gradual approach to generalization in order to help them overcome the difficulties they will have to face in later grades. In the following we will present a first 'inventory' of the situations we have identified, subdivided according to the ambits they refer to: linguistic, perceptive, social, mathematical.

A1. Generalization and language: the role of argumentation

The students of a class (11 years old), who are used to argumentation, are exploring a growing pattern, whose components are called 'pyramids', with the aim of identifying general laws to connect the characteristics of every pyramid (the total number of triangles it contains, the number of rows, the number of white triangles…) with its position in the pattern.

When the class is working to find a general law to determine the number of black triangles in the row which constituted the base of every pyramid, a student (Y.) observes: "*On the line where the pyramids lie … for example, in the fourth pyramid the black triangles are four and the white are three … my pyramid of six floors has six black triangles and five white triangles on its base… The white (triangles) are always one less than the blak ones. Maybe a pyramid with any number of floors has a number of black triangles on its base which is equal to the number of floors and as many white triangles as the black ones minus one*". The teacher of the class proposed this reflection as a comment to the transcript: "Before her intervention, Y. wasn't aware of her conclusions but, as she was verbalizing, she started deducing and expressing the general rule".

This example highlights the fundamental role played by the relationship between argumentation and generalization in the social construction of knowledge. This relationship can be made explicit only when argumentation becomes a shared

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[4] In the ArAl Project, as a linguistic mediator, we use Brioshi, a virtual Japanese student who doesn't speak the Italian language but knows how to express himself using a correct mathematical language. Brioshi is an algebraic pen friend with whom students communicate using mathematical sentences which should be written through a correct application of syntactical rules in order to be understandable (Malara e Navarra, 2001).

tool for the teacher and the students: every component of the class has to get involved in this process and has to relate him/herself with the ways in which the other components get involved. This means that the students must take the responsibility for their learning and that the teacher must take the responsibility for fostering students' social construction of their knowledge.

We could say that the power of argumentation is related to the fact that those who start developing it are not completely aware of their ideas before they try to express them. As argumentation becomes an habit, the student understands its value and becomes aware of its role in comparing facts and in making their similarities gradually emerge, together with their connecting thread.

A2. Generalization and language: the *potential general*

Through the activity called 'pyramids of numbers' (the sum of every couple of numbers written on two adyacent bricks is equal to the number on the brick over them), the teacher guides students toward the identification of the law which expresses how to determine, without any calculation, the number written on the brick at the top of a three-floors pyramid as a function of the numbers written on the three bricks on the basis of the same pyramid.

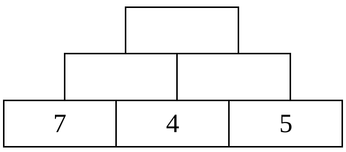


Fig.2a

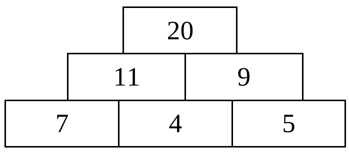


Fig.2b

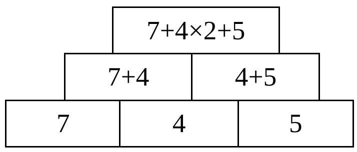


Fig.2c

The classical method of completion (Fig.2b) is not enough in order to determine the required law because it leads to an 'inexpressive' result (in this case 20). The non canonical representations (Fig.2c), instead, allow the construction of what we call a *relational-ontological* representation of the number at the top of the pyramid, i.e. the representation which constitutes the best explicitation of the general law "The number at the top is the sum of the two side numbers and the double of the middle one". The next step to be carried out is the translation of the equality $20=7+4\times2+5$ into natural language. The final step for students is becoming aware that this sentence, expressed in natural language, constitutes a *potential general* through which it is possible to carry out a further conversion into algebraic language: $n=a+2b+c$. We think that the first,epistemological, source of difficulties associated with the use of letters in mathematics, is related to the capability of *conceiving a letter as a number*. This aspect could represent an insurmountable barrier to algebraic language and generalization.

The concept of *potential general* could be related to the notions of quasi-variable (Fuji and Stephens 2001) and *quasi-generalization* (Cooper and Warren, 2011) as possible bridges between arithmetic and algebra for students from 6 to 14 years old. This observation leads to the introduction of an other theoretical construct, essential in the construction of the necessary conceptual and methodological premises in an effective approach to generalization.

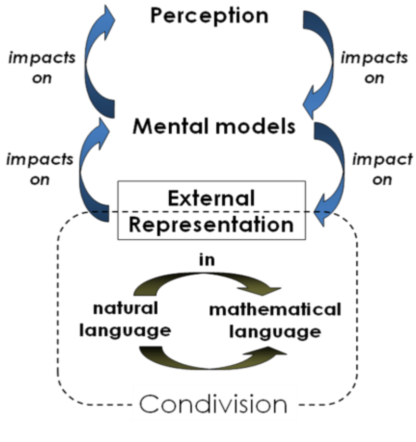
A3. Generalization and language: the pupil as *thought producer*

The ‘law’ identified in the previous example of the bricks pyramid is: “The number on the top is the sum of the two side numbers and the double of the middle one”. This conclusion represents an important moment of condensation in the evolution of algebraic babbling. The pupils have been guided towards the collective construction of a general, though improvable, definition and have formulated its explicitation. They were protagonists as *producers of ‘original’ mathematical thought*: it means that they were able to express with a clear and synthetic language what they have understood and what they have said in public. Traditionally, however, the teacher is the one who mediates between the topical moments of institutional mathematical thinking (principles, theorems, properties, etc.) and their application; in these cases the pupils are mainly *re-producers* of a theory, to the organisation of which they are basically strangers. On the contrary, it is very important that pupils are educated – through forms of collective exploration of thought-provoking problematic situations – in *producing, in the natural language, general conclusions to be shared with the classmates and the teacher*, organising them in a coherent and communicable way, as an intermediate step towards a later *translation* into mathematical language.

B. Generalization and perception

Perception, i.e. the psychic process operating a synthesis of sensory data into meaningful forms, developed in a socio-costructivistic context, allows to create meaningful premises to the approach to generalisation. If, for example, one is asked to express his/her calculation strategies in order to find out the number of pearls contained in this necklace:

two different perceptions arise, which lead to two different representations of the counting strategies (on this aspect, see also paragraph D1): (a) visualising the black and the white pearls separately leads to the representation 2×9+3×9; (b) ‘concentrating’ on the pattern leads to (2+3)×9. We interpret the dynamics of the situation in the classroom through the following model:



If an (a) or a (b) pupil were alone, he/she would limit him/herself to his/her personal mental model and to its consequent external representation, because he would not be motivated towards searching for other interpretations, and therefore counting modes. A didactic contract based onto collective argumentation, on the contrary, promotes the sharing of knowledge: each pupil compares his/her representation with the other one and discovers that his/her way of 'seeing' the necklace is not the only one. The result is therefore a feedback that influences the internal representations and the new way in which the necklace structure can be perceived. The social construction of knowledge promotes the evolution of thought towards a shared conquering of new meanings. Overcoming the initial difficulty of integrating the other's vision is the first step towards the understanding of the equivalence of the representations: $2\times9+3\times9=(2+3)\times9$. This shall lead to the development of the general meaning of the equality $a\times c+b\times c=(a+b)\times c$ and therefore to the understanding of the distributive property (Malara & Navarra 2009).

C. Generalisation and conceptualisation: the conceptual condensation

The class (10-years-old) is exploring the behaviour of a scales, seen as a metaphor of first grade equations at one unknown quantity.

Teacher: Let's describe the situation.

Jacopo: On the right hand side there was baking soda and 100 grams. On the left hand side there were three glasses of baking soda.

Teacher: And what are we aiming at?

Jacopo: We want to find out how much a glass of baking soda weights.

Teacher: Ok. So what have we done, Matteo?

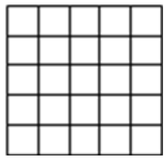
Matteo: We have removed a glass from both sides, then we have divided by two the content of both dishes. So now we have a glass of baking soda on the left and 50 grams on the right. A glass weights 50 grams.

We refer to the transition from the *dynamic* phase of concrete, generative activities, which characterize the pupils' educational path particularly in the first eight years of schooling, to a phase in which the teacher promotes the condensation into knowledge of the mathematical concepts underlying the activities. The one in the example is meant to promote the need to spot out the principles of equivalence as tools to represent the experiences carried out. These new concepts shall then be linked to knowledge concerning operations on natural and relative numbers, to the properties, to the use of letters, to the meaning of 'equal to'. By reflecting onto the experiences carried out, the pupils are guided towards the identification of general principles that allow to solve other, structurally similar situations. A weak leading in this transition phase does not allow – and sometimes inhibits – the progressive approach to generalization, since the pupils shall keep operating at a concrete level, without working out any theory.

D1. Generalisation and foundational mathematical aspects: the evolution of counting strategies

During our cooperation between Italian classes of the ArAl project and English classes (pupils aged 9 to 15) we presented the following situation:

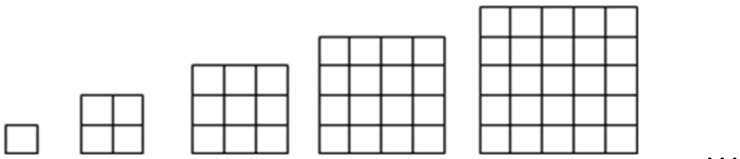
> This drawing represents a structure made of toothpicks.
> Count the number of toothpicks and explain in the mathematical language your counting strategy. It doesn't matter to determine the number of toothpicks.



With the Italian pupils (13 years old) we discussed the strategies produced by the English pupils (15 years old): (i) 5+5×11; (ii) 3×(3×5+1)+6+6; (iii) 5×4+5×4×2. We asked them to interpret these strategies so as to make clear the meaning of these expressions. The evident result was that each counting strategy reflected the way in which the groups had perceived the structure of the construction (see paragraph B). For instance: the Italian pupils explained that the members of the group (i) had seen the five pillars as 'combs', and they had then added the last five vertical toothpicks. Free to count, the pupils discovered many alternative strategies, some of which were more 'economical' than others. When they were guided in comparing the expressions, they found out equivalences through proves, e.g. for (i) and (iii):

$$5+5\times11=5\times4+5\times4\times2 \rightarrow 5\times1+5\times11=5\times4+5\times8 \rightarrow 5\times(1+11)=5\times(4+8) \rightarrow 5\times12=5\times12$$

Starting from this activity, generalization arises as soon as the *static* situation is transformed into a *dynamic* one, that is in the moment in which students begin to explore how the counting strategies change in relation to the changing of the square's dimensions, and they are asked to say if it is possible to find out a 'law' that allows to determine the number of toothpicks that are necessary to build a given shape. The pupils discover that it is better to organize an in-order research, for instance through a display of drawings of the following kind:



1 2 3 4 5

The pupils are guided to activate a common counting strategy which express the interrelation between the number of toothpicks and the number of the place of the corresponding square and which can be expressed through a formal representation of the number of toothpicks of a construction, at the generic place n. In this way, they can identify the structures that allow to express the relations connecting the numbers in play in a given problematic situation, i.e. its *structure*. In this case (if n is the number expressing the position and s is the corresponding number of toothpicks) they write, for instance, $s=2n(n+1)$. If the teacher concentrates mainly on the calculus processes, neglecting the reflection on them, she prevents the pupils from going through the experience that is necessary to the process of generalization and to the conceptualisation of arithmetical structures.

D2. Generalization and foundational mathematical aspects: the progressive achievement of the concept of structural analogy

Rosa (kindergarden - 5 years old) is comparing cardboard ‘trains’, the carriages of which contain objects set in a precise order. She is concentrating on two of them.

Teacher: Why are you looking at those two particular trains? What do they contain?

Rosa: Here is a red, a red and a yellow.

Teacher: Yes, they are Duplo bricks. And what have you got in this one?

Rosa: A walnut, a walnut, a sunflower and it goes on so.

Teacher: So what?

Rosa: They are almost the same.

In this example, Rosa is doing algebra, since she finds out in a naive way the *structural analogy* between the two trains. Right from kindergarten or primary school, pupils can be allowed to recognize relationships between the elements of a sequence and their place number. They discover analogies (in this case, between two train structures), describe them with words and represent them with a code (e.g.: AAB), thus approaching a germ of formalised language, and therefore generalization. The common construction of the code, developed at the stage allowed by the pupils age, hence represents the *collective* result of a *relational* reading of the situation, in which the attention is concentrated not on its elements, but rather on the relationships that connect them. Being able to spot out such correspondences between different situations allows the development of *analogical* thought. Kindergarten constitutes the first step of this process, within a logic of continuity with primary school, where these germs of thought shall gradually ripen along the following school grades, through the exploration of a kind of arithmetic built up in the perspective of the development of algebraic thinking, hence towards a more mature generalization and a more advanced kind of abstraction.

Conclusion

What we have described shows educational aspects that we believe should be constantly strengthened, since they support the process towards generalization, promoting in the pupils *metalinguistic* and *metacognitive* aspects, and consequently reflection: (A) on *language*: the ability to construct argumentations, to translate from natural into algebraic language, to produce original thought; (B) on the relationships between *perception* and the social construction of shared knowledge; (C) on passing from *concrete generative situations* to the construction of concepts (conceptual condensation); (D) on some foundational mathematical aspects: the evolution of counting strategies and the progressive attainment of the concept of structural analogy.

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