ArAl: a Project for an Early Approach to Algebraic Thinking

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International research in mathematics education, and in particular regarding algebraic teaching/ learning and its difficulties, – at diverse ages from junior levels through to university – have underlined a widespread traditional teaching method quandary. Over the past twenty years, research has focalized on a large number of possible approaches that increment the *meaning* of the algebraic *processes and objects*. One of the principal forms is: the *problem solving* (where emphasis is given to the analysis of problems and equations), the *functional* approach (the use of letters to indicate measurement and the formal coding of relations among measurements), the *generalization* approach (the use of expressions to represent geometric patterns, numerical sequences, 'rules').
A determining role is attributed to the *linguistic* approach and to research that faces the didactical developments starting from the *concept of algebra as a language.* This role becomes even more significant if it is associated to *the hypothesis of an early approach to algebraic education beginning from the didactical revision of the relations between arithmetic and algebra.* Research does demonstrate just how the students' limited arithmetic experience becomes an obstacle when learning algebra. It is thought that an earlier approach can reduce this difficulty (see for instance Kieran 1992, Linchevski 1995).
It is only recently that interest has been shown towards an early approach to algebra (see Da Rocha Falcao & al. 2000, Carpenter & Franke 2001, Carraher & al. 2001, Kaput & Blanton 2001, Lee 2001, Malara 1999, Schliemann A. & al. 2001). Question and answers are being formulated, such as: a) How *early* should *early algebra* be? b) What are the *advantages and disadvantages* of an anticipated start? c) How are the answers to these questions connected to *theory* of cognitive development and learning, and to the cultural and education t*raditions* of teaching algebra? d) Which algebra and algebraic thinking aspects should be part of an early algebraic education? e) What consequences would an early algebraic education have on *teachers and their training*?
This final question summarizes one of the most crucial aspects connected to the implementation of innovative didactical sequences as those of the ArAl Project.

Links between arithmetic and algebra
The ArAl project is situated within that theoretic frame that assumes the denomination of *early algebra*: in which it is thought that the principle cognitive obstacles are to be found in the pre-algebraic field, and that many of these spring up from unsuspected arithmetic contexts and they then become conceptual obstacles to the development of algebraic thinking. Numerous recent studies in the field demonstrate how students lack appropriate arithmetic structures from which they can generalize, and moreover, how students lack the knowledge of arithmetic procedures and do not posses a conceptual base from which to build up their algebraic knowledge.
The didactical problems regarding elementary algebra can be identified at the construction level of: (a) basic arithmetic knowledge; (b) algebraic knowledge.
The first level (which corresponds roughly to the ages of between 6 to 12 year olds) does not give sufficient attention to the passage to algebra; the second level (traditionally around the age of 13) tends to concentrate excessively on the calculation processes. The result being that algebraic thinking is not constructed progressively as a thought tool *parallel* to arithmetic, but s*uccessive* to arithmetic, thus above all its manipulative mechanisms and computational aspects are highlighted. Therefore algebra loses some of its essential characteristics: one, an appropriate language to describe reality and two, a potent reasoning and forecast instrument of codifying through formulas knowledge (or hypotheses) regarding phenomena (in our case elementary) and where new knowledge derives (by means of transformation consented by algebraic formalism) on the phenomena themselves knowledge (or hypotheses) regarding phenomena (in our case elementary) and where *new knowledge derives from* (by means of transformation consented by algebraic formalism) on the phenomena themselves.
Let us now follow a reverse path. We will begin by proposing some reflections about didactics of Algebra in secondary schools (11-14 year olds) and then we will climb back up along

the branches of an imaginary genealogy tree towards the 'arithmetic ancestors' of algebraic concepts.

Potentially misleading models

As mentioned previously, traditional teaching is going through a crisis; further to the reasons listed initially, we believe that two probable causes of didactical and psychological nature are:

- investing too much time in technique exercises,
- the lack of recognizing psychological and cognitive barriers that impede the students' acceptance of the algebraic language.

International research among 12-13 year olds responds that elementary algebraic notions are not necessarily difficult, however the defects can be found in the didactical practice which does not take sufficiently into consideration:

a) a widespread inadequacy of arithmetic comprehension,
b) linguistic difficulties connected to learning a formal language.

We shall discuss the linguistic difficulties later on, here we would like to point out some examples of how legitimate models relating to operations acquired in an arithmetic ambient may be mis-leading or inhibiting to the conceptual progress of an algebraic ambient.

Some research begins from the consideration that the model of multiplication as a repeated addition learnt at elementary school implies that *multiplying and multiplier* are *whole* numbers, for example: the student 'sees' '5 + 5 + 5 + 5' as '5 x 4', read as '5 repeated 4 times'. Later, however at an algebraic level, if the writing '3x' refers to that model, and therefore is interpreted as '3 repeated x times', many students lose track of the meaning in front of that '3' repeated '*how many* times?' due to the fact that students cannot 'see' the number of times (Navarra 2001). On the other hand, if the student is capable of interpreting '3x' as 'x + x + x', that is 'x repeated 3 times' then the passage of a not whole multiplier may form yet again a logical passage that is difficult to grasp: in '0,3x' repeating x *for 0,3 times* is senseless because it does not have a comforting or concrete support. Even knowing from arithmetic that commutative property holds for multiplication, students often see multiplying and multiplier as things having a *different status*. For example, in the algebraic field, in '2y' they see '2' as a *different* entity from 'y'. Also because, although they are able to grasp in '2y' the commutative property and therefore the equivalence between 'two times y' and 'y times 2', if they write 'y2' the teacher will tell them they are wrong – thus consolidating their misconceptions in that which could be defined as a diverse ontology between the *number* and the *letter* (of course it is of fundamental importance that the concept of convention is studied in-depth).

Natural Language and Formal Language

Difficulties like these, within an arithmetic field, then influence that long chain of possible errors that students encounter when they face setting up an equation of a problematic situation. For example students: 1. attempt (just as an *naive* translator would) to 'literally' translate the text; 2. do not know or do not use the algebraic notation conventions; 3. interpret number as adjectives, and letters as labels or as abbreviations; 4. interpret an equation as a sequence of instructions, in which case the sign '=' means 'give place to'; 5. do not know how to interpret the texts of 'non-sequential translation' problems, meaning problems in which the order of the terms used in the text is not satisfactory to their mathematical elaboration; 6. do not clearly distinguish the sums and powers produced (ambiguity between additive and multiplicative structures); 7. have confused ideas about ratio and difference.

It is believed that unconscious habits and cognitive process present within a natural language may create conflict with the procedures required from a formal language. For example: 'y is three times bigger than z' is *literally* translated erroneously as 'y = 3 x + z' ('three times more than z') or 'y = 3 x > z' ('three times bigger than z'). In other words, it is presumed that without a complete awareness of arithmetic procedures and writings, students possess an impoverished conceptual base which impedes their future construction of algebraic knowledge.

However, it is opportune to underline that often, students' errors and misconceptions are neither stupid nor lighthearted and they represent a result of reflection and reasonable attempts to attribute a meaning to mathematical expressions that would otherwise lack significance. Others could indicate reasoning one might define, as not as being wrong rather as being *interrupted* and could therefore represent the beginning of a potentially productive reasoning.

2. AIM OF THE PROJECT

A collective construction of meanings

The abovementioned consideration about the presence of a potentially productive reasoning brings us to what we have written previously about early learning algebra. Some youngsters – above all the younger ones at elementary school ages – are less conditioned by errors and stereotypes and express themselves more creatively and are more willing to amuse themselves. Thus within the class they can be lead to a collective construction of new *meanings* through the practice of reflections, interpretation hypotheses, 'murky' language use, which are aspects often destined to remaining in the limbo of '*the unsaid'* thus creating errors and misconceptions which hinder the students' relationship with mathematics and more in general their relationship with school.

On a linguistic level, some of the major difficulties that younger students have to face with algebra, are represented by having to understand:

- *why* a symbolic language has to be used;
- which *rules* does the symbolic language have to abide to;
- the difference between *solving* and *representing* a problem.

The perspective of initiating students to algebra as a language, within a continual backwards and forwarding use of arithmetic thinking, may favour the individualization of a more effective didactics with students aged between six and fourteen, as it is based on *negotiating* and thus on the *explicitness* of a didactical contract aimed at the solution of algebraic problems based on the principle *"first represent and then solve"*. This perspective (developed in depth further on) seems promising when facing one of the most important conceptual areas of algebra: the transformation of *representation* terms from the natural language in which they are formulated into the formal algebraic language translating the relations that they contain. In this way the search for the solution is transferred to the next phase.

Before facing the duality of represent/solve, one must concentrate on a fundamental point of the theoretical frame that the ArAl project refers to.

Algebraic babbling

We retain that there is a huge similarity between learning a natural language and learning an algebraic language; so, as to explain this point of view we have adopted the *babbling* metaphor.

When a child is learning a language he/she slowly approaches its meanings and rules and gradually develops these through imitation and use until school, when the child then learns to read and reflect upon the grammar and syntax aspects of the language. In traditional didactics of algebraic language one begins by firstly studying the *rules*, as if formal manipulation came before the comprehension of meanings. There is the tendency to teach algebraic syntax however at the same time its semantics are overlooked. Mental models of Algebraic thinking should instead be built up through what we call initial forms of *algebraic babbling*. Our hypotheses is that algebraic thinking and mental models of thought should begin from the first years of elementary school – years in which pupils begin to encounter arithmetic thinking, making it possible to teach them to *think about arithmetic in an algebraic way*. In other words, building up *progressively* in students algebraic thinking as an instrument and object of thought closely *interwoven with arithmetic.* Starting from its *meanings* and by means of constructing an environment that informally stimulates an autonomous elaboration of *algebraic babbling*, thus favouring the experimental approach to a new language in which the rules position themselves gradually, and within a didactical contract which tolerates initial 'promiscuous' syntactical moments.

Solve and Represent: product and process

These considerations lead us to a delicate area of construction on behalf of the students and their *ideas* about mathematics, ideas which participate in the formation of that which Schoenfeld refers to as the students' *epistemology*. Referring to what he/she is convinced about thus leading him/her retaining that the solution to a problem (a simple addition for elementary students or a more complex problem for an older student) is essentially – or exclusively -: *the search for a result.* This of course moves the students' concentration towards that which can produce said result, that being the *operation.* Solving problems basically means *calculating*. Students should be aided into thinking of how to distance him/herself from the worries of the result and consequently of the operations that will permit reaching that result. In this way, they should be aided into reaching a higher level of thinking: substituting the act of *calcula-*

ting with 'looking at oneself' while calculating. It is the passage from a cognitive level to a meta cognitive one in which one has to *interpret the structure of the problem.*
We can say that the first case aims at the individualization of the *product*, that is the operations that consent one to *solve* the problem – whereas the second example of the problem aims at the individualization of the *process*, that being the writings which permit one to *represent* the manifestation of an articulate thought process. In the first case the *diachronic* aspect prevails: the mental processes of calculation take place sequentially in time and the result emerges at the end of an *action*. In the second case the time dimension disappears: the author abstains from *doing* and poses his/her interest in the conceptual dimension of individualizing the *structure* that the algorithm applied.
This is a basic concept to understanding the passage from an *arithmetic* way of thinking to an *algebraic* way of thinking. This is a very delicate step because it is linked to one of the most important aspects of the epistemological gap between arithmetic and algebra concerning the explicit and implicit contracts supporting the two procedures: whilst arithmetic requires an *immediate* search of a solution, on the contrary algebra *postpones* the search of a solution and begins with a formal trans-positioning from the dominion of a natural language to a specific system of *representation.*
In our opinion and as we have underlined previously, the perspective of algebra as a language can enhance the individualization of a more efficient didactics with students aged between seven and fourteen, due to the fact that it is based on *negotiating* and therefore *explicating* a didactical contract aimed at solving algebraic problems based on the principal *"first represent, then solve"*. A promising perspective when facing one of the most demanding and important areas of conceptual algebra: *the transposition in terms of natural language representation in which problems are formulated and described, into the formal algebraic language in which first the relations they contain are translated and then their solutions are found.*
We feel that profound changes are necessary within algebraic teaching spheres at lower secondary school, and that starting from elementary school it is opportune to *anticipate the approach to these problems. This can be done beginning from the individualization of didactical concepts and processes which favour the passage from arithmetic thinking to algebraic thinking.*

3. METHODOLOGICAL ASPECTS

The ArAl project and teachers

What we have written so far synthesizes the frame which contains the ArAl project's activities. Until this moment (July 2002) the ArAl project's main users are teachers of elementary and secondary school (6-14 years old pupils) who, in general, do not have a mathematical university background, as the majority come from areas of humanistic and pedagogical education (elementary teachers) and scientific education (secondary school teachers).
Hence, the project presents itself as an important occasion for teachers to reflect upon their *knowledge* (which, of course, conditions the choice of modalities through which teachers themselves then transmit their knowledge to students) and their beliefs regarding mathematics – one could say regarding their *epistemology.* The situations laid down in each unit take place in stimulating didactical environments – which can often be difficult to handle – and require various competences and numerous delicate capabilities on behalf of teachers. In other words, teachers who intend facing innovative didactics must be prepared to encounter, as mentioned previously, his/her knowledge, competence and beliefs together with a mix of methodological and organizational aspects. These are *not at all secondary aspects,* in fact they operatively support the actual *culture of change.*

Some significant aspects

The didactical contract

Constant check up on the clearness of the didactical contract in all of its phases, especially in elementary school. This means that the objective of the project is not that of supplying technical competence in advance (for example, through Unit 6 '*From the scales to the equations*' it does not intend teaching how to solve a first grade equation). The objective is to investigate which are the more adapt forms for building up mathematical concepts in students that will help them towards a gradual formation of algebraic thinking.

Students must be made to understand what is the essence of the didactical contract: that they are the *prime characters* in the collective construction of *algebraic babbling*. This means e-ducating them gradually towards even complex forms of a new language by favouring their reflections on the differences and equivalences of mathematical writings and its meanings – a gradual discovery of the use of letters instead of numbers – the application of properties – the understanding of the different meanings of equal sign – the infinitive representations of a number, and so on.

Mathematical discussion

Activating a collective discussion about mathematical themes leads to privileging *meta-cognitive and meta-linguistic* aspects; students are led to reflect upon language, knowledge and processes (solving a problem and translating it into algebraic language). They also have to face hypotheses and their classmates' proposals, to compare and classify translations, to eva-luate their own beliefs and to apply responsible choices. Thus teachers must be aware of the *'risks'* and of the particularities of this type of teaching method. Discussion helps to incre-ment potential in *thinking arithmetic with an algebraic key*, and research has highlighted just how much *verbalization and argumentation* are fundamental vehicles for understanding.

Protocol interpretation

Building up competences for a refined interpretation and the successive classification of stu-dents' proposals and protocols means being faced with an enormous variety of mathematical writings, which are often elaborated with a mixed and personalized use of language and symbols which have been put together more or less correctly. This behaviour is developed well if the teachers themselves stimulate creativity as well as reflection. When students reali-ze they are the *producers of mathematical thinking* and are contributing to a collective con-struction of knowledge and languages, they express a huge variety of proposals, many of which are far from being banal and which, when put together, represent a common patrimony of all the class. This is the important point where the teacher needs the ability to pick out (and to let students pick out) the paraphrases of a possibly correct sentence by selecting the wrong, ambiguous, bizarre, translations. These are important activities as they help not only the stu-dents, but moreover the teacher, in understanding that every text in whatever mathematical context can be read and interpreted at different levels, even due to the organization and for-mulation of the text in a natural language.

4. THE UNITS

Unit 1: The Brioshi Project (7 – 12 year olds)

The Unit privileges the approach to *linguistic* aspects of mathematics. They are developed a-round an imaginary character, Brioshi, a Japanese student who can only communicate using the mathematical language, and who enjoys exchanging problems and solutions with classes of other nations. The units propose *translation* activities from natural language to arithmetic language and vice versa, starting from simple phrases like 'From 4 take away 2' and progres-sing to more complex activities like the 'Game of the hidden number' ('To a hidden number add four to obtain ten'). This Unit demonstrates how the exchange of messages may begin by traditional tools (simulations, notes, faxes) until it reaches the 'mathematical communication' of two classes (contemporarily by means of a chat line set up using *MSN Messenger Service* software).

Unit 2: The numbers chart-grid (7 – 14 year olds)

This Unit represents a gym for pre-algebraic thinking through to actually being the area of first grade equation application. Activities are developed around the exploration of a square of one hundred number boxes from 0 to 99. Through the discovery of regularity, and games us-ing 'numeric pathways' within and on fragments of the grid ('Treasure island', 'The island game', 'Never never land'), therefore problem situations even on grids of diverse dimension and reflections upon the different ways to represent numbers in the boxes, the Unit leads to generalization by using letters and thus at last to the 'conquest' of the grid having a dimension of n x n

Unit 3: The Numbers Pyramid (6 – 14 year olds)

The Unit intend to favour the development of *relational* thinking. By exploring the 'pyramids' made up of 3, 6, 10 bricks, students are led to individualizing and representing a more and more complex link among the numbers written within the bricks. Emphasis is given to the bi-nary aspects of the operations and the *non-canonical* representation of the numbers. At the start the activity takes place within an arithmetic ambient and then progressively widens towards algebra and the *naive* discovery of the use of letters and equations. Reflections on the representation helps to highlight the linguistic and meta-linguistic aspects.

Unit 4: Matematóca & other mathematical games (7 – 8 year olds)

In this Unit by means of original variants of board games (Dominoes, Memory, Bingo) or by means of invented games (The masks game) students are supplied with material that obliges them to re-visit arithmetic arguments from a view point that favours an algebraic vision. At the same time, by using opportune didactical mediators (smudges, clouds, slips of paper, etc.,) students approach the unknown number and the possibilities of the ways of representing it. Step by step as the games proceed, the materials that make up their concrete supports modify and the indications written in natural language are transformed into simple algebraic writings in which the unknown is represented by the score of the dice used in the game.

Unit 5: Regularities (10 – 11 year olds)

In this Unit the activities involve the need to discover the regularity of a structure. In the first stage, students analyze necklaces made up of different coloured beads which are positioned alternately in the necklaces; in the second stage, students analyze structures that are made up of matches that form houses, bridges and nets of various dimensions; in the third stage, they analyze friezes and stamps; in the fourth arithmetic sequences. At each stage, through exploration and discussion, students search for regularities and successively for the representation in mathematical language. The discovery of regularities is precious in forming pre-algebraic thinking, as it favours the passage to generalization.

Unit 6: From the scales to the equations (10 – 14 year olds)

This Unit approaches algebraic thinking. Through collective solutions to problematic situations and with the pan scales students discover 'the principle of equilibrium' and the two principles of equivalence; the passage from an experimental activity through to its written representation leads to the 'discovery' of letters in mathematics and equations. Even algorithms for the solution to equations are progressively elaborated and refined through collective and individual activities during which students elaborate and compare diverse representations, refine their competence of natural language translations and symbolic ones and vice versa and moreover students get used to using letters as the unknown. A succession of opportunely organized verbal problems of different levels of difficulty lead students to investigating how to solve problems using algebra.

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