

POP-UP Workshop

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1. Pop-up books

Pop-up books (*action book, livre animé, libro animato*) should ideally be part of every child's experience and in every library for young people. During the 18th Century these books were produced specifically to amuse children by the ingenious incorporation of things to move and that moved.

The first true pop-up was *Little Red Riding Hood* and was published in 1855 in London. Gradually German publishers came to dominate the rapidly developing techniques of colour printing and book production.

In 1925 Josef Albers and Laszlo Moholy-Nagy in the Bauhaus began to attend to mechanisms by which paper can be transformed into three dimensional objects.

Today pop-up books are still planned essentially *to amuse*, but in recent years these techniques have been used to produce pop-up books with serious educational aims.

2. Pop-up paper engineering

Pop-up paper engineering is the set of techniques based on paper cutting and folding which create 2-dimensional mechanisms located within books, cards, dioramas that become three-dimensional shapes when they are opened. The techniques can be analysed geometrically.

Fundamental to pop-ups is the *process of transformation*. The pop-up object is drawn on paper and it becomes three-dimensional by making appropriate cuts and folds. Between the two phases, there is *movement*. The result is extraordinarily different from the starting-point not only because of the third dimension but also because it is obtained in a *dynamic* way. It is surprising because it is sudden and unexpected.

The *imagination* is essential in linking the plan and the pop-up and supporting the conscious anticipation of the outcome of the transformation from one to the other. One must learn *to visualise* on the paper something which at that moment exists only *in the mind*.

Errors are inevitable but generally they are self-evident and the correction easily made. Even the youngest pupils realize when what they have obtained is not what they had in their mind and spontaneously experiment further to gain success.

3. *A dynamic geometry*

A pop-up is realized in three phases: (i) prepare a plan on a sheet of paper or cardboard, (ii) prepare the individual cuts and folds; (iii) cut and fold the sheet in the appropriate places.

A pop-up designer needs to be able *to imagine the object three-dimensionally before making it*, which means s/he can anticipate the necessary cuts and folds in the lines of the plan.

Practical experimentation, consideration of mistakes and subsequent discovery of general principles are essential parts of the process of pop-up design. One must anticipate what parts of the paper will become horizontal planes and what vertical ones, what will be foreground and what background, what upper and what lower, what high and what large, and so on *before* realizing the pop-up.

From planar geometry to spatial geometry and vice versa

Pop-ups offer the teacher a *dynamic geometry* teaching experience through planning and executing conceptually challenging practical activities. From a general point of view, pop-ups can support *the passage from planar geometry to spatial geometry* and vice versa.

Depending on the geometric principles considered, every pop-up mechanism can be analysed at increasing levels of complexity. Exploring the mechanism enables the realisation of *working plans which move in parallel with the base plans out of which they have been obtained by cutting and folding. Cuts are parallel one another as well as the folds. Cuts and folds are perpendicular to one another.*

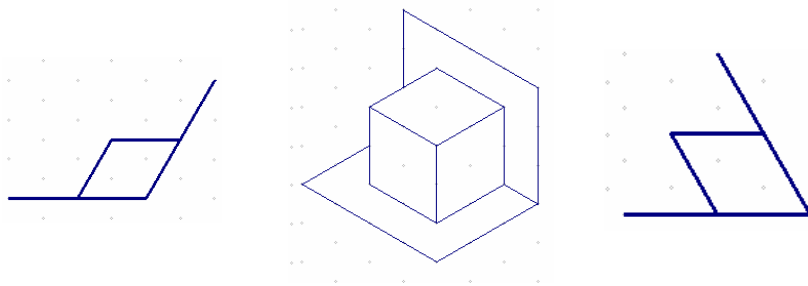
Teaching mathematics with pop-ups may include all of these aspects:

- *investigating* the mechanisms,
- *predicting* the results of cutting and folding,
- *verifying* that general principles are indeed true,
- learning from mistakes,
- creating novel pop-ups.

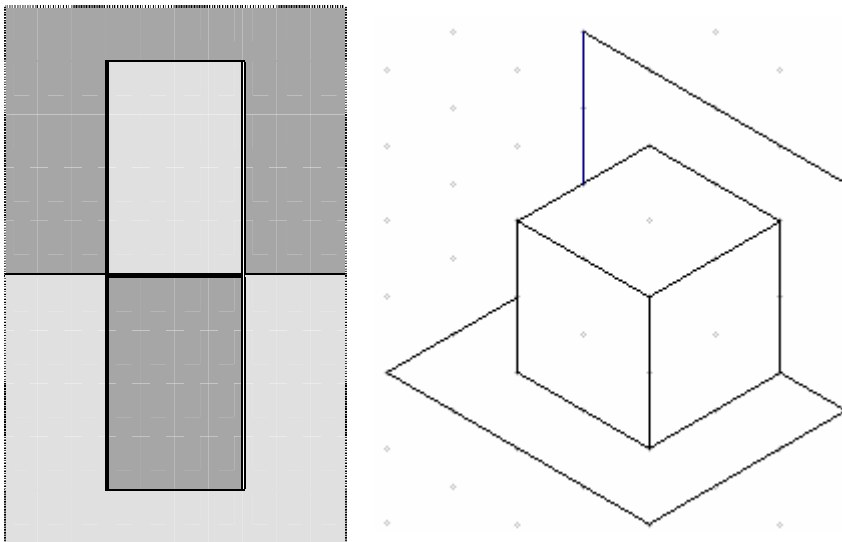
Naturally the age of the pupils and the aims of the teacher may mean that only some aspects are addressed. The aesthetic and expressive aspects are often motivating factors for pupils.

4. A possible teaching programme for pop-ups

1. The teacher shows a pop-up cube and slowly opens and closes it. S/he asks the students: How do you make a similar one?

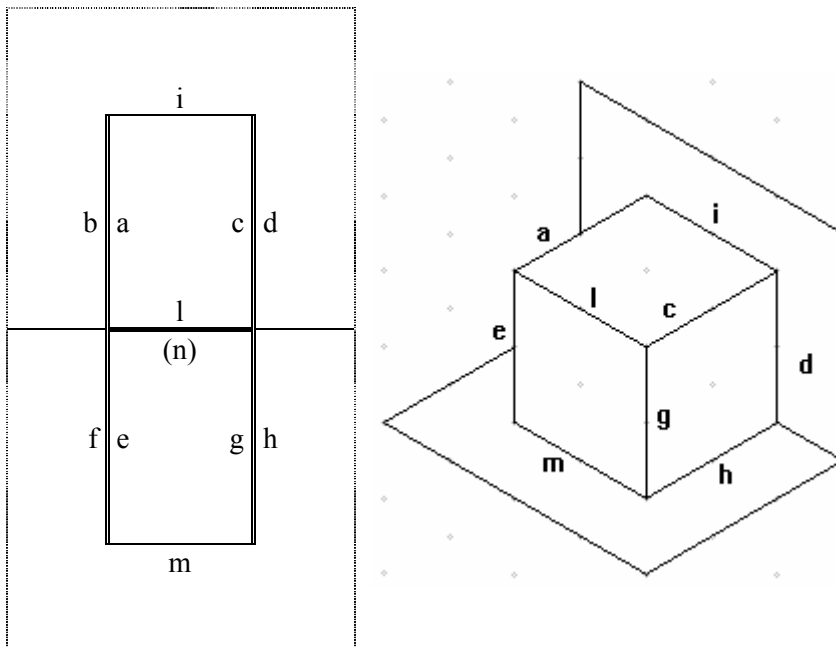


2. Plan a pop-up cube on a piece of flat, plain paper



Pupils work on the pop-up design (the key is the same as for origami: a thin line for 'valley' or 'book' folds, a thick line for 'mountain' or 'roof' folds, a double line for cuts). The teacher shows some projects and asks pupils to interpret them and to describe what one would obtain by folding and cutting the paper.²⁰

3. Explore the relations between the lines on the plan and the edges of the pop-up cube.



Pupils work with the pop-up cube. They think about the segments, angles, parallel, perpendicular and skew lines and use instruments (set-square, protractor, compass).

²⁰ To assist the reader, in some pop-up plans the parallel faces have the same colour; moreover, almost always an isometric projection of the realized object is drawn near the plan. In every paragraph a short commentary is written in italic script.

Pupils can represent the results of their research in a table like the following, containing parallel ($//$), perpendicular (\perp) and skew (S) edges of the pop-up cube.

a													
b	\perp												
c	$//$	S											
d	S	$//$	\perp										
e	\perp	$//$	S	$//$									
f	$//$	\perp	$//$	S	\perp								
g	S	\perp	\perp	$//$	$//$	\perp							
h	$//$	S	$//$	\perp	S	$//$	\perp						
i	\perp	\perp	\perp	\perp	S	S	S	S					
l	\perp	S	\perp	S	\perp	\perp	\perp	S	\perp				
m	S	S	S	S	\perp	\perp	\perp	\perp	$//$	$//$			
n	S	\perp	S	\perp	S	\perp	S	\perp	$//$	$//$	$//$		
	a	b	c	d	e	f	g	h	i	l	m	n	

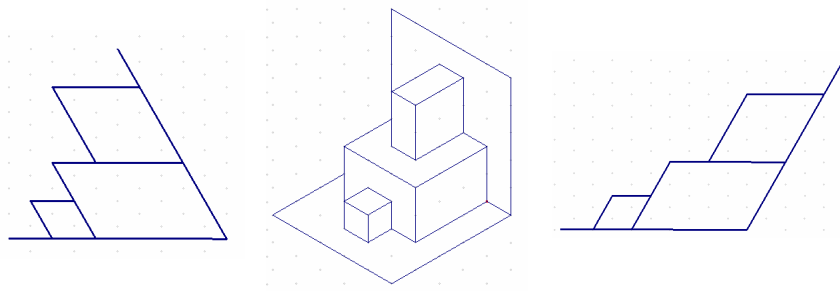
4. Explore the relationship between the pop-up cube and a 'true' cube.

Pupils begin by investigating a pop-up cube and comparing it with a 'true' cube, noticing similarities and differences (pop-up solids are as incomplete as those words in which letters are half-erased and are Gestalt - recognizable by reorganizing the field). They discover that there are 2 faces out of 6, 11 edges out of 12 and all 8 vertices are visible.

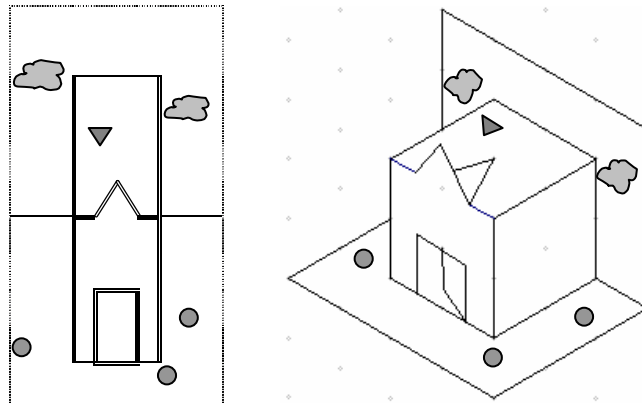
The pop-up cube is the starting-point for successive explorations during which one discovers that cuts and folds can be modified and that solids made from superimposed cubes can be built. The unlimited expressive chances of pop-ups come out of these discoveries.

5. What other geometric properties can be explored with the pop-up cube and why?

When the paper is not fully open the three-dimensional objects are parallelepipeds rather than cubes. What properties do these shapes have? What is the relationship between their faces



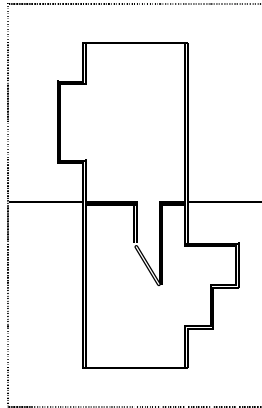
6. Below is the plan for a pop-up house. Add clouds in the sky, flowers in the garden, trees in the distance, a bird on the roof etc.



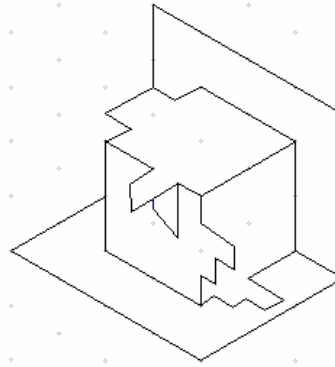
The pupils learn to recognize in the plan drawing (the project) the three dimensional object and to verify their conjectures. They must decide on the project – before cutting and folding – what remains horizontal and what becomes vertical when they realize the pop-up.

7. Before cutting and folding imagine what these pop-ups will look like for the following two plans. After having realized them, compare your hypothesis with the pop-ups realized and reflect on your possible mistakes.

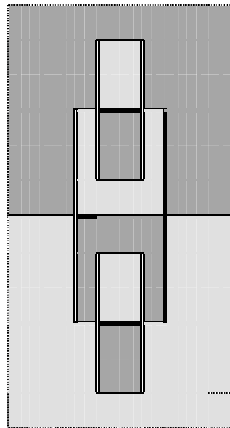
Pop-up 1



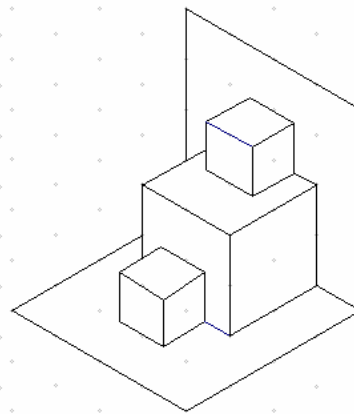
Pop-up realized



Pop-up 2



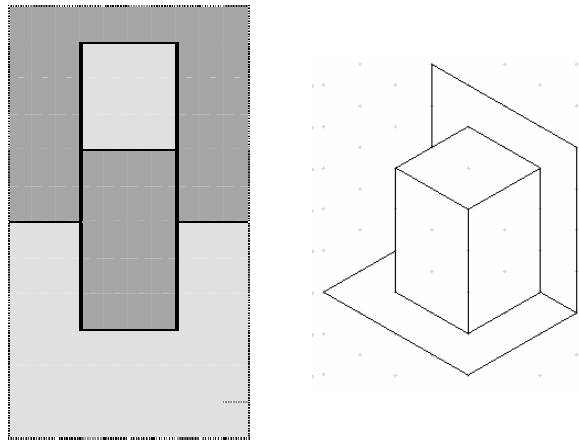
Pop-up realized



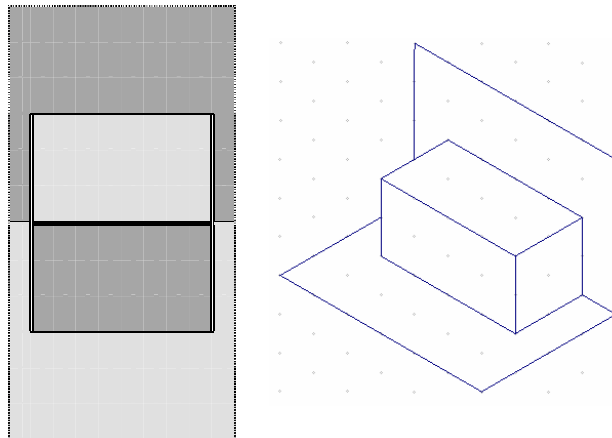
Pupils can easily notice, realizing the two pop-ups, that both protrude from the paper. This is a topic in pop-up engineering and will be further explored in the next paragraphs. Before doing it we will consider another pop-up.

8. Plan and make a pop-up cuboid (cuboid of type A)

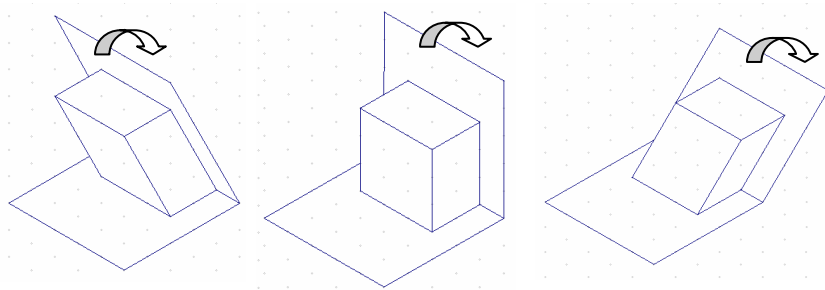
Pupils continue exploring the passage from the pop-up cube to the cuboid and analyse how altering the dimensions of the edges effect the fold lines (the hill fold no longer agrees with the main fold of the sheet and this represents, for many, a logical obstacle to overcome).



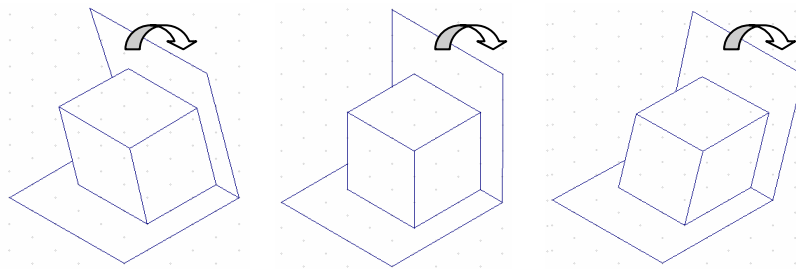
9. An other kind of pop-up cuboid (type B)



10. The dynamic aspect of the pop-up technique helps pupils to see that a cuboid is simply a special case of prisms which have a parallelogram as their base...

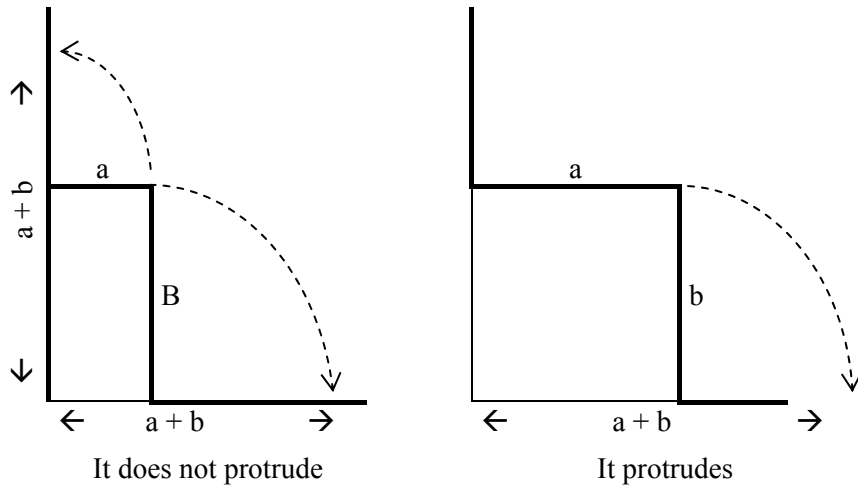


... and the cube is an equilateral prism having a rhombic base.



11. And now: *To protrude or not to protrude? It's a nice question.*

Question A: Under what conditions will a pop-up not protrude when folded?

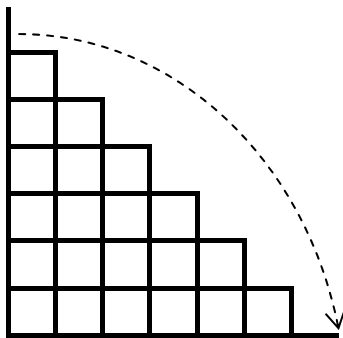


$$a + b \leq p/2$$

The sum of the dimensions of the pop-up must be less than or equal to half the sheet.

12. In the first case the cuboid does not protrude.

Can you plan other non-protruding cuboids on the same sheet?

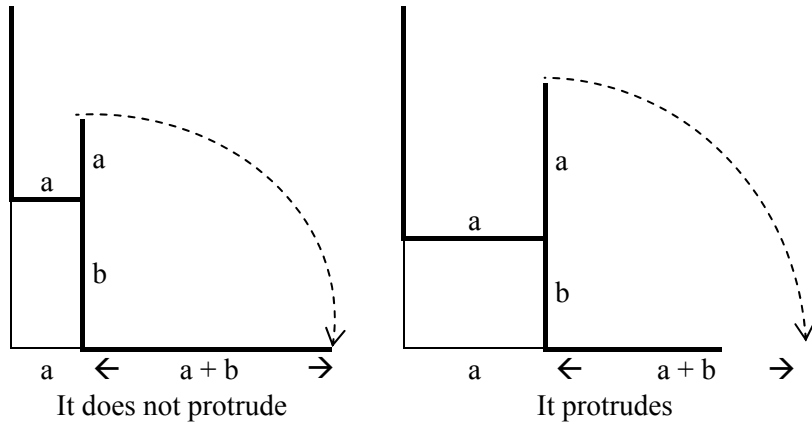


The solution is: all the rectangles having their half-perimeter equal to half the length of the sheet generate non-protruding cuboids. In the example (only natural numbers):

<i>a</i>	1	2	3	4	5	6
<i>b</i>	6	5	4	3	2	1
<i>a + b</i>	7	7	7	7	7	7

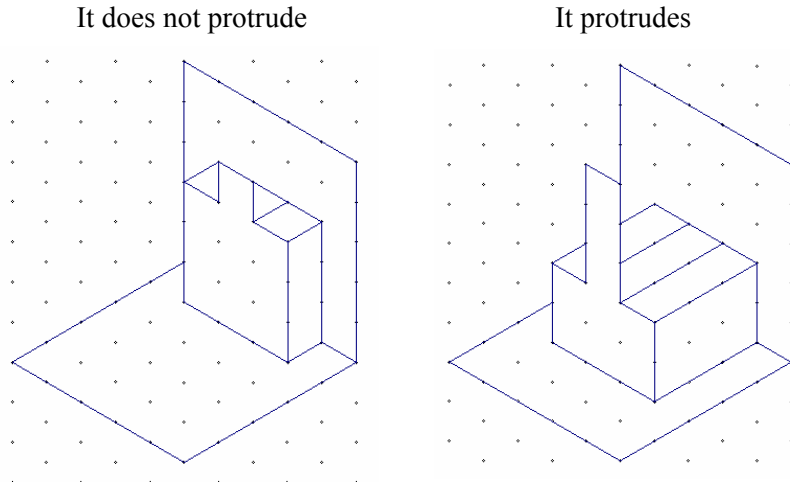
13. To protrude or not to protrude?

Question B: Under what conditions will a pop-up not protrude when folded?



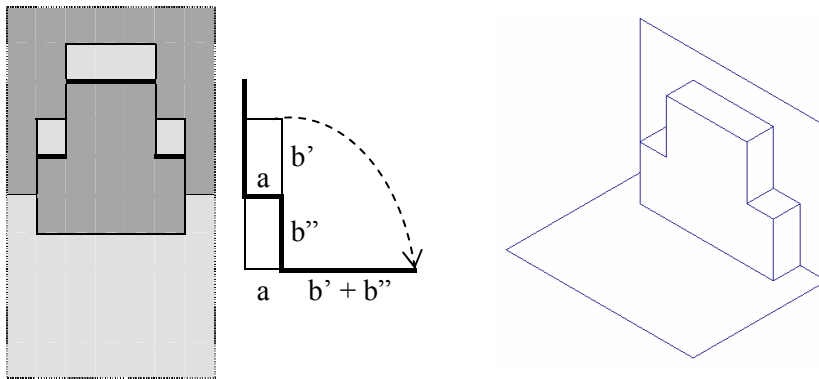
$$a + (a + b) \leq p/2 \rightarrow 2a + b \leq p/2$$

In this case the sum of the double depth $2a$ of the pop-up and its height $a + b$ must be less than or equal to half sheet (see the pop-up solid of paragraph 6).



14. *To protrude or not to protrude? Question C:*

Under what conditions will a pop-up not protrude when folded?

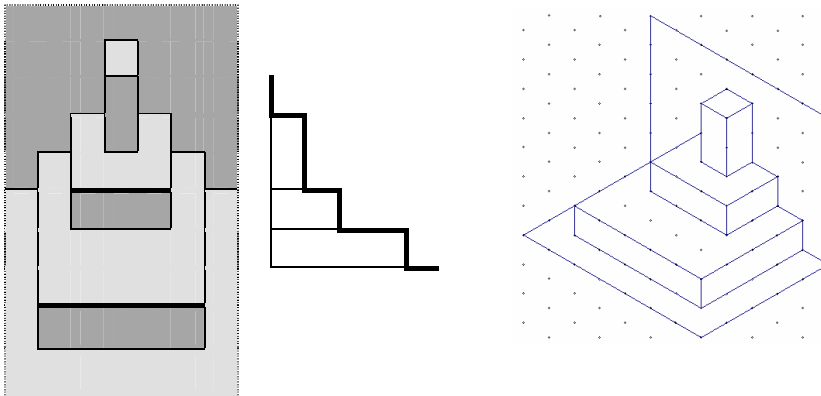


$$a + b' + b'' \leq p/2$$

The sum of the depth a of pop-up and its height $b' + b''$ must be less than or equal to half the sheet (see pop-up solid of paragraph 8).

15. *To protrude or not to protrude? Question D:*

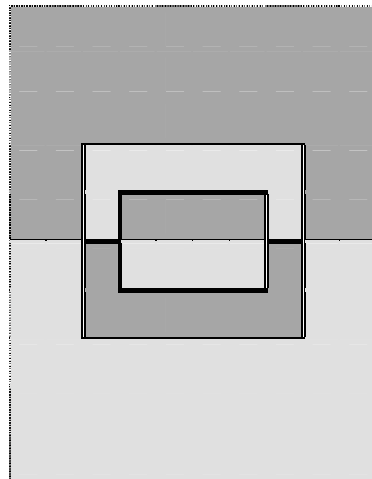
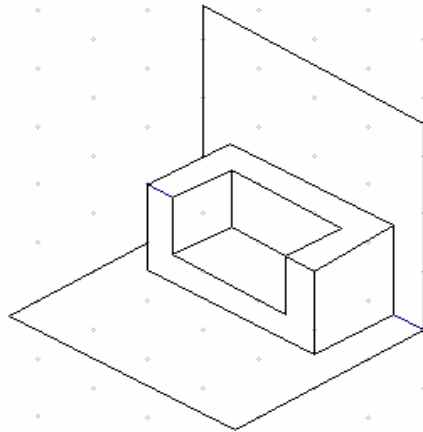
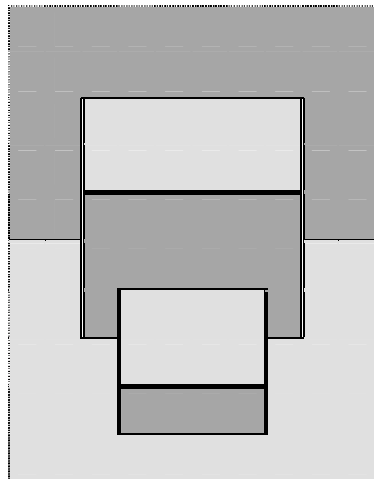
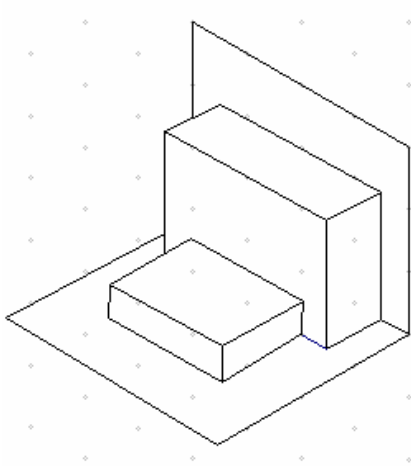
Under what conditions will a pop-up not protrude when folded?



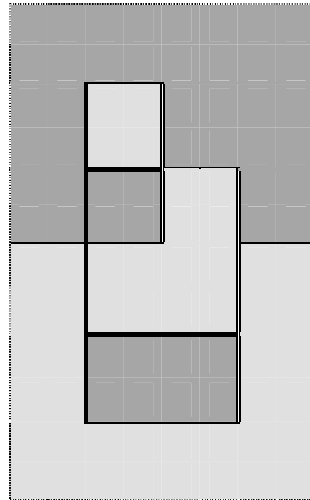
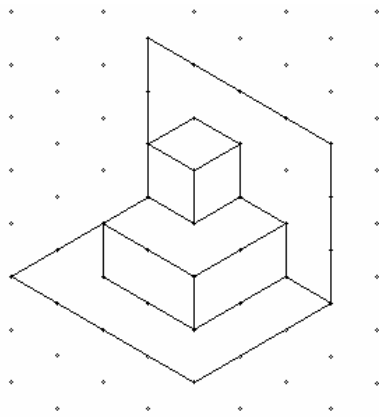
$$a_n + b_n \leq p/2$$

The sum of the depth and the height $a_n + b_n$ of every cuboid of the pop-up 'pyramid' must be less than or equal to half the sheet.

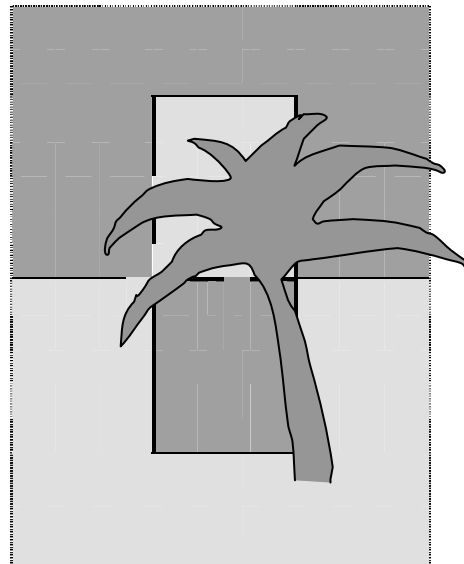
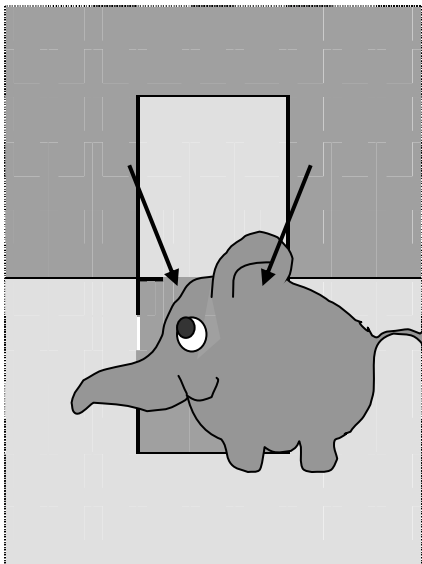
16. Plan and make pop-ups of the following shapes drawn on isometric paper.



17. Design your own pop-up on isometric paper, plan and make it. For instance:



18. Will these pop-ups work when cut and folded?



The elephant pop-up works, the palm tree pop-up doesn't work.

The pop-up cube is the technical base of both drawings. One must pay attention to let almost one link between the front face and the upper face of the cube (in the elephant pop-up the two links are pointed out by two arrows).

In the second case, if one tries to realize it and cuts along the outline of the palm leaves, this breaks every link between the two faces and the pop-up doesn't work.

19. Pop-up extensions

- Look at some pop-up books and cards. How do they work?
- Do all pop-ups require parallel folds?

Can these pop-ups be realized? Can they succeed? Do all of them succeed, or some do and some others do not?

Try to *imagine* it, predict the result. Then, verify your conjecture.

Fig.1

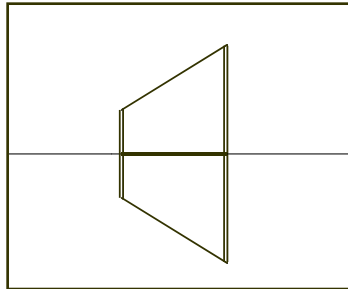


Fig.2

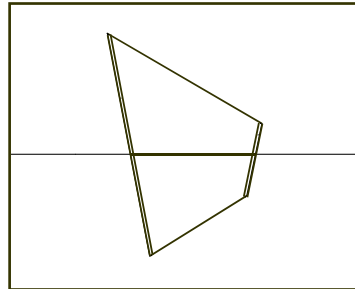


Fig.3

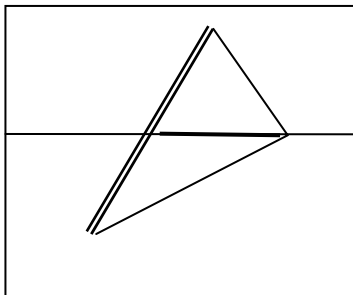


Fig.4

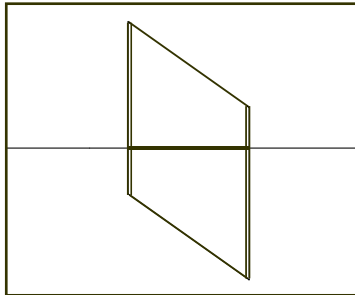


Fig.5

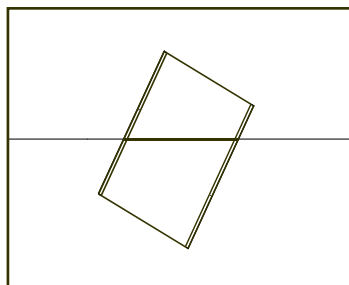
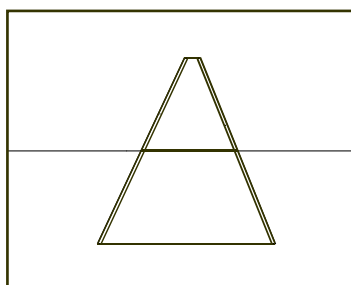


Fig.6



The exploration of these figures can represent for older students the beginning of a very stimulating investigation in the pop-up world in which you find symmetries, angles, bisectors.

Cabri geometry will be your useful travelling companion.

- c) How do you construct plans of pop-ups accurately on plain paper?
 In order to construct pop-ups accurately pupils will need to work with geometric equipment accurately e.g. rulers, pairs of compasses, possibly protractors. It is also possible to plan pop-ups using computer drawing packages such as Draw, Cabri or CAD-CAM.
- d) Work with teachers from art and design to see how you can exploit pop-ups.

The potential here is boundless. You could also work with language teachers to design pop-up story books.

5. Conclusions

Making a pop-up book that has no protruding pop-ups can be a teaching goal for a programme of teaching which includes the discovery of the *principles by which figures perform movements that are consistent with the size of the book.*

Pupils can discover these principles for themselves by solving practical problems and challenges. When errors are evident it is possible to correct them by observation, reasoning, sensible use of mathematical knowledge, manipulation, and the motivation and enjoyment of discovery.

Pop-ups may improve pupil's capabilities in relating two and three dimensional situations. Being powerful on a *logical* level and involving

on *expressive* level, it is appropriate for being explored within mathematics (at growing levels of complexity appropriate to the age of the pupils) and being utilized as an instrument in multi-disciplinary activities.

In order to fully realise the potential of pop-ups for mathematical activity and logical development the teacher needs to encourage pupils to think for themselves. Pupils need to be given the opportunity to establish the 'rules' for themselves and to be encouraged to explain why these rules work and need to be adhered to.

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