APPROACHING THE DISTRIBUTIVE LAW WITH YOUNG PUPILS

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This paper contributes to the research strand concerning early algebra and focuses on the distributive law. It reports on a study involving pupils aged 8 to 10, engaging in the solution of problem situations, purposefully designed and presented through concrete objects, drawings, oral or written descriptions. The study focuses on ways in which perception *leads to different mental images that influence the choice of either the $(a+b)\times c$ or the $(a)\times c)+(b)\times c)$* representation*. Our hypothesis is that understanding these dynamics is a fundamental step for the construction of a meaningful approach to properties based on suitable activities, organised so as to favour an explicit statement of proposed solutions and a collective comparison of arithmetic expressions that codify solution processes.*

1. Introduction

This work is part of the ArAl project, which was designed to revisit arithmetic teaching in a pre-algebraic perspective (Malara & Navarra, 2001, 2003a, 2003b) and concerns a fragment of a teaching path centred on problem solving activities finalised to construct in pupils an experiential basis for an objectification of the distributive law, through collective discussions for sharing and reflection[1]. The distributive law, together with associative and commutative laws, plays a key role on both the arithmetical (mental calculations, algorithms, rule of signs, …) and algebraic side (transformation of expressions, recognition of equivalence relationships, formal identities, …) and more generally in the production of thinking via algebraic language. In usual teaching practice however, these properties are taken for granted, almost assumed as tacit axioms, or worse, they are assigned to be learned by heart from the textbook. Pupils are thus led in the position not to understand the sense of these properties, to perceive a rupture between the experiential and the theoretical, and not to recognise their value on the operative level. The tacit spreading of this phenomenon is documented by studies concerning teachers (Tirosh *et al.*, 1991) and by studies focusing on a conscious learning of arithmetical properties and of the distributive law in particular (Mok 1996, Vermeulen *et al*. 1996). In our project this property enters the game in many situations and it is exactly due to this pervasiveness that we deemed important to design a path aimed at its objectification through problem situations that highlight its genesis. Our first results highlight the influence of perception on the construction of mental images, useful for conceptualising the property, and the effectiveness of processes of sharing.

2. The situation

The class[2] which is object of the present analysis, is beginning a path which will lead to the conceptual embryo of the distributive law.

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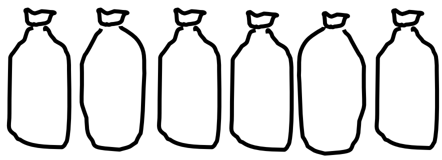
[1] The theoretical frame of the work is essentially the one of the project and it is sketched in the quoted papers. The English version of the project is in <**www.matematica.unimo.it/0attività/Formazione/ArAl**>**.**

[2] It is a grade 4 class from Birbano (Belluno, Italy) at the beginning of the year school (2002-03). The activity was planned within a yearly cycle of meetings in which the teacher researcher Giancarlo Navarra and Cosetta Vedana, class teacher for the mathematical-scientific area were simultaneously present.

The objective is to construct premises for subsequent developments, finalised to the appropriation of the property as a *mathematical object.* The activity develops through three problem situations that favour the development of dynamics that can be summarised in three phases: 1) from confusion to the first arithmetical representations; 2) from the first perceptions to the two constitutive representations of the property; 3) reflection on the two representations and appropriation of the mutual equality of the expression values. The three situations are meant to favour the transparency of the transition from *perception* of the situation to *translation* into mathematical language. It is thus necessary to (a) *educate* pupils' perception, i.e. lead them to become aware of the existence of *diverse* ways to perceive a situation, among which some may be more *productive* from a mathematical point of view; (b) make pupils understand that it is possible – through collective sharing – to understand the meaning of translations and conceptualise their mutual equivalence beyond the process each of them identifies. Very often teachers themselves must be educated analogously.
We present here a teaching sequence, overall lasting about three hours (distributed in three sessions) to be considered as an example of the evolution of thinking in both *individual* and *collective* forms. The most meaningful parts of the diary are described in detail, whereas other parts, meaningful for their overall sense, are synthesised. As the reader will notice, the initial interventions by pupils denote a certain confusion about the assigned task and an apparent regression with respect to competencies that were acquired the previous year in the solution of problems that had a similar structure. These are consequences of the assignment, repeatedly asking *not* to *solve* the problem finding a *result*, but to *explore* one's own *modus operandi*. Confusion is thus due to an atypical didactical contract: pupils are asked to work at metacognitive level and this request, although having strong educational value, is harder to be managed by both students and teacher. One of the main features of the ArAl project is to favour reflection on processes: to obtain this, it promotes activities that stimulate metacognitive and metalinguistic competencies and construct sensitivity towards these aspects in teachers.

3. Phase 1: from confusion to the first arithmetical representations

(i) Problem presentation and assignment

The teacher puts 6 bags (made of a non-transparent fabric) on a desk and explains that each of them contains 7 triangles and 12 squares.	

The task is *to count how many objects are in the bags totally*. It is strongly underlined that the important thing is *not the number* of objects, but rather the *reasoning process followed* to find it. Pupils know the quantities referred to objects but cannot see them, therefore they are forced to construct *mental models*. In order to do this, they must initially *focus on their perception* of the imagined situation, in an intertwining of *unstable* perceptions and *floating* calculation attempts.

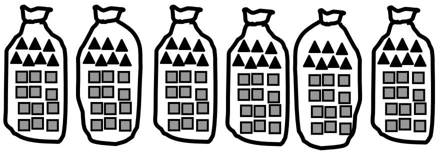
The task is complex: in fact the pupil is asked not to *count*, but to *look at himself/herself in the counting act*. He/she must face a metacognitive task: reflecting on his own actions.

3.1. From confusion …

(ii) (Class[3]) The first difficulty is a psychological one: pupils are anxious and do not understand the task.

(iii) (C) The next step occurs at cognitive level: being uncertain about the task, pupils go for the most familiar interpretation and mentally count the content of a bag: there are 19 items[4]. These are in a single bag, but they show to be thinking that anyway one step is completed, because what seems to be important for them is the *number*, i.e. the 'result'.

(iv) (TR) A *visual* aid is given: pupils are invited to open the bags.



(v) (C) Still *confusion*: seeing the objects does not seem to offer significant help. While searching for an interpretation of the task, pupils start manipulating the items: they group them by colour, by shape, others leave them shuffled. However, this manipulation does not provide particular hints.

(vi) (TR) *The task is reformulated and a discussion is solicited*: "I did not ask you to tell me a number …do you remember? I asked you to count mentally the pieces and then try to explain how you proceeded for counting. Look inside yourselves, as in a movie. What did you think? Where did you start from?"

(vii) (C) The new discussion is still confused, but more choral and animated, with weak metacognitive features: some pupils say that they count pieces *one by one*[5], others *using the times 2 - table*, others using the *times 5 – table* (discussion about the strategies highlights that they count pieces two or five at a time, to go faster), others by *groups of colours*.

(viii) (TR) The *task is formulated again*, and pupils are invited to give less generic explanations, *taking into account the information given by the problem*: "Look carefully at what is in front of you: there are *six* bags; each bag has the same content, made of triangles and squares, and there are *seven* triangles and *twelve* squares. Your brains are working with *these* numbers".

(ix) (C) The activity evolves at *metacognitive* level: pupils, working in small groups, manipulate blocks meditating on the moves, with slow shifts accompanied by reflection; in a Gestaltian sense, pupils are *restructuring their field*, searching for

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[3] From now onwards C will stand for 'class' and TR for 'teacher researcher'.

[4] Pupils are still not seeing the items, but need to imagine them, hence 'counting' is done on a virtual context, without the reassuring feedback given by a physical contact with objects. But, as we will see later, pupils will keep interpreting the request to 'count' in a strict sense, an operative one, instead of management of a complex situation in which 'counting' may become a sort of umbrella, under which several strategies for calculation can be developed.

[5] As underlined in previous note, 'counting' still emerges as a litany.

meaningful perceptions. Calculation processes start shaping up in a complete and communicable way. Embryos of processes are proposed: for instance, pupils of a group say that in order to find the total number they “did 19 times 6”.

Steps i – ix:
The initial situation (i) in which triangles and squares are not visible makes pupils uncomfortable (ii) and is sorted out by means of a calculation *(iii) but it forces pupils to construct mental images of the situation.* Seeing *physically the objects (iv) does not help in the beginning (v) because possibly the real problem does not lie in vision per se, but in the* organisation *of the vision itself. The repeated invitation to* look inside oneself *(vi-viii) leads to an increasing development of* metacognitive *activity and, consequently, to the elaboration of more organised attempts to ‘see’ the situation with the eyes of mind. Hence a virtuous circle is enacted (ix) between an increasingly ‘guided’* perception *and a growing clarity in the interior visualisation of* mental processes *and in their* verbal description. *The situation is mature for Brioshi’s*[6] *entry.*

3.2. … to the first arithmetical representations

(x) *Mathematical language* enters the scene: it is time to verify if and how field restructuring – and hence the game of back-and-forths between perception and development of mental models – has produced images that can be represented through mathematical language. *Brioshi* is called in: (TR) “What message could you send him to explain how you *managed to count triangles and squares inside a bag* and then *to find* the total number of *triangles and squares*?”

(xi) (C) Part of the pupils formulate (individually) the following proposals, transcribed on the blackboard and then discussed.

(a)	**$9 + 7 + 3 = 19$**
(b)	**19×6**
(c)	**$(5 + 5 + 5 + 5 + 5)$**[7]
(d)	**$5 \times 3 + 4 = 19$**
(e)	**$5 \times 4 - 1$**
(f)	**$2 \times 4 = 8 + 11 = 19$**

Sentences highlight a *short circuit* with the task. Except for (b), the others express a conviction that different ways of expressing the content of a bag, that is 19, must be listed. This misunderstanding leads to substantially unreasonable

[6] Brioshi is an imaginary Japanese pupil (variably aged according to the age of his interlocutors) and is a powerful support within the ArAl project (the first Unit is completely dedicated to him). He was introduced to make pupils aged between 7 and 14 approach formal coding and a difficult related concept: the need to respect rules in the use of language, need which is even stronger when engaging with a formalised language, because of the extreme synthetic nature of the symbols used in them. Brioshi is able to communicate only through a correct use of mathematical language and enjoys exchanging problems and solutions with foreign classes, through a wide range of instruments, such as messages written on paper sheets or more sophisticated exchanges through the Internet.

[7] The proposal comes by a pupil with difficulties.

expressions, often impenetrable, because the authors cannot justify the reasons underlying their representation.[8]

(xii) (TR) *Description of the situation*: inviting pupils to use a representation in mathematical language was premature and natural language becomes again the mediator – with a fundamental role, given the age of pupils – through which pupils are asked to *describe* the *concrete* situation as it is.

(C) At the end of the discussion, the class comes to a collective formulation: "There are six bags, all on a desk: there are 7 triangles in each bag and 12 squares in each bag, we must represent and find how many they are altogether".

(xiii) A proposal of sending a new message to Brioshi is made, in order to take into account what has been said.

(xiv) The class formulates different proposals, showing an evolution with respect to the previous ones:

(g) $7 + 12 = 19$
(h) $7 \times 6 + 12 \times 6$
(i) $72 + 42$
(j) $19 + 19 + 19 + 19 + 19 + 19$

Through discussion pupils focus on (h), (i) and (j) but they see them as *different* things. They do not grasp the underlying mental models.

(xv) (TR) Pupils are asked to re-*describe* the situation in written natural language.

(xvi) (C) Some descriptions are still generic, for instance: "there are 6 bags and 2 different shapes", but two families of descriptions emerge that mark the beginning of a *turning point*: (a) "6 groups of squares and 6 groups of triangles"; (b) "6 bags, in each bag there are 7 triangles e 12 squares".

(xvii) (TR) Pupils are asked to write other sentences for Brioshi individually; two groups of sentences come out, referring to the two models:

(A) a x c + b x c	**(k)**	**7×6 6×12 $72 + 42$**
	(l)	**$72 + 42$**
	(m)	**$6 \times 12 + 6 \times 7$**
(B) (a + b) x c	**(n)**	**19×6**
	(o)	**$12 + 7 \times 6$**
	(p)	**$(12 + 7) \times 6$**
	(q)	**$7 + 12 \times 6$**

(xviii)(TR) Pupils are asked to comment on the formulations.

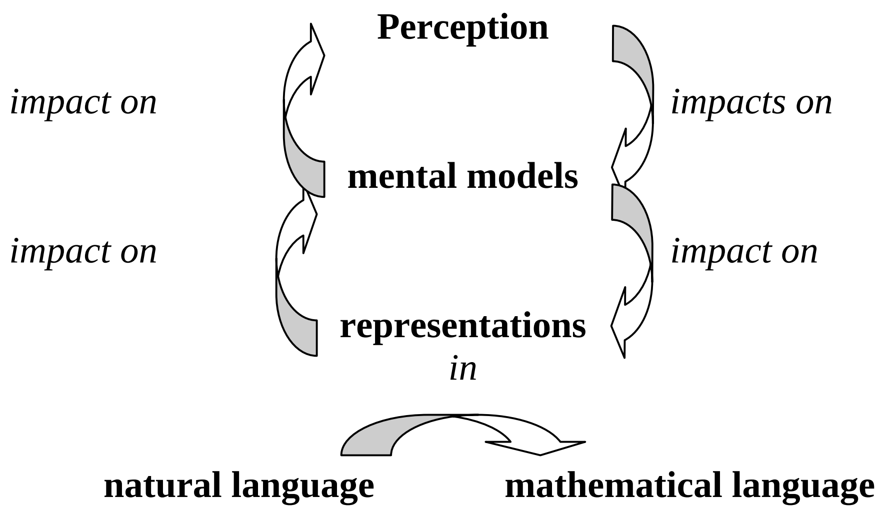
(xix) (C) At the end of discussion these conclusions are reached:

[8] It often happens that when the task is not clearly understood, pupils that express a higher self-confidence are the least aware whereas more prudent pupils show to have a stronger critical capacity and prefer to 'stay at the window'.

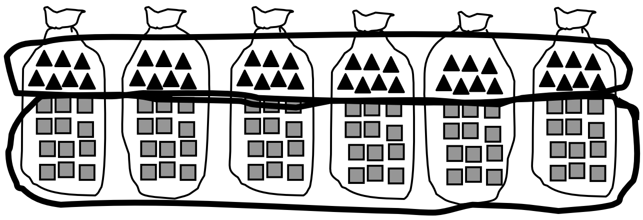
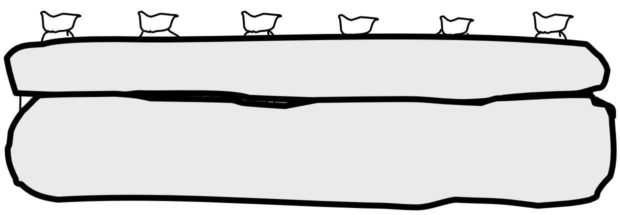
Group (A) conclusions'*… first they find the whole lot of triangles and then the whole lot of squares';* group (B) conclusions say that '*… they calculate the number of squares and triangles altogether'.*

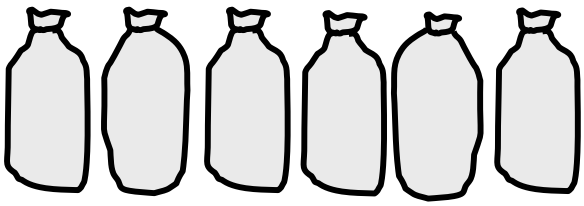
Steps x – xix

Brioshi's entry (x) starts up an activity of representation in mathematical language; after a start influenced by a possible misunderstanding on the task (xi) the recourse to formalised and natural language alternatively (xii-xiv) produces increasingly meaningful results. A system of relationships is outlined that can be visualised through the following model:



Pupils' increasing capability in moving inside the relationships illustrated in the model leads to the production of sentences (i), (j) and (k) in (xiv), the transparency of which makes possible to trace back the organisation of perceptions that generated them. Pupils are the protagonists of this reconstruction, through which the activity is read at a metacognitive level.

(i) $7 \times 6 + 12 \times 6$ The relationship between representation and the mental model elaborating perception is *transparent*.	
(j) 72 + 42 The representation (R) reflects the previous mental model (M) but it is *opaque*. The relationship between (M) and (R) is weak. Control at metacognitive level is scarce.	

(k) 19 + 19 + 19 + 19 + 19 + 19 The representation refers to a perceptive act, and therefore to a mental model, that although different is still *opaque*.	

Through further intertwining of natural language (xv-xvi), mathematical language (xvii) and natural language again (xix), pupils reach conclusions that introduce effectively an embryo of the distributive law (xix).

4. Improving the two representations

(xx) (TR) A new problem is proposed (a week later):

> Granny prepared for Santa Lucia 8 bags of sweets for her nephews.
> In each bag she put 5 chocolates and 14 candies.
> How many sweets did granny buy? [9]

(xxi) (C) Pupils solve it with no questions about clarification. Two types of solutions are provided and they can be ascribed to both representations:

(A) a x c + b x c (5 pupils)	**(r) $5 \times 8 = 40$** **(s) $14 \times 8 = 112$** **(t) $112 + 40 = 152$**
(B) (a + b) x c (6 pupils)	**(u) $14 + 5 = 19$** **(v) $19 \times 8 = 152$**

Models (A) e (B) are nearly equally distributed; proposed calculations are all carried out separately until the result is obtained.

(xxii) (C) During the discussion two pupils provide decisive contributions:

(A_1) Denise wrote in a rough copy:

$$5 \times 8 + 14 \times 8 =$$

but she did not know how to continue and preferred to go back to single operations. She explains she recognised the same problem she tackled previously.

(B_1) Giada realises that Denise’s procedure is the ‘translation’ of the first solution type and tries to translate the second type solution, writing:

$$14 + 5 + 19 \times 8$$

but she realises that it is not good. Collective discussion helps her to modify it:

$$(14 + 5) \times 8.$$

Steps xx – xxii

Presentation of a new problem (xx) raises two types of representations by separate steps, which can be reduced to those of distributive law (xxi); during discussion representations in a line appear (xxii). The former representations are blocking, whereas the latter constitute a fertile *ground.*

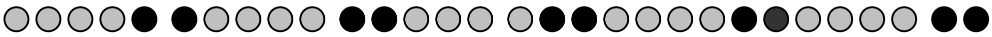
[9] The question ‘How many sweets …’ although focusing on the outcome and not on making the process explicit is nevertheless clear to pupils due to the established contract.

Leading pupils to representations in a line *seems to be a necessary condition (although not a sufficient one) to construct a mental attitude that may favour the transition to an embryonic view of the property. As we said earlier, this condition is subordinated to an* education to perception *of elements of the problem situation. At a first level, most pupils are attracted by aesthetic, formal and expressive aspects that distract them from the logico-mathematical aspects. Denise and Giada are probably two among the few students that show a natural inclination for selective analysis. Generally,* education *plays a determinant role: this means leading the class, through sharing, to make perceptions and reasoning explicit, so that differences may become productive for a collective construction of shared knowledge.*

5. Reflecting on the two arithmetical representations

(xxiii)(TR) A week later, a third problem situation is proposed:

A giant cardboard necklace made of alternating four grey beads and two black beads is shown:



The task is the usual one: to explain in either natural language or mathematical language (or both) the *way* in which one can find how many beads compose the necklace.

Again pupils must try to *describe* what they are *thinking*.

(xxiv)(C) Proposals are compared and commented upon:

(Giulia) I count how many the beads are: 2 × 5 + 4 × 5
but 'how' did you count? [note written by the teacher]: I counted this way: the beads are thirty and to make this result I counted them with multiplication.[10]

(Lorena) I calculate how many the black beads and the grey beads are: 2 × 5 + 4 × 5[11]

(Claudia) Every two black beads there are four grey beads[12]: 2 × 5 + 4 × 5

(Giada) Two black ones and then four grey ones[13]: 2 × 5 + 4 × 5

(Alberto) (2 × 5) + (4 × 5) = 10 + 20 = 30
I did 2 the number of black beads and I multiplied it by 5, same thing for four.

The class realises that everybody used formulations of a single type (A)[14].

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[10] For many young pupils the verb 'to count' has a similar meaning to the verb 'to calculate'. Perhaps to Giulia the two verbs express the same action, the same content, and this action and content can be expressed only in mathematical language, or rather: to her the latter is the most 'spontaneous' way to find the number of beads. The activity is carried out at cognitive and not metacognitive level.

[11] Lorena suggests that, once explained what *she does*, numbers and mathematical signs express *how you calculate*.

[12] This is the description of what she sees in the necklace: is it also the "description" of the way in which she gets to the solution?

[13] See previous note.

[14] The pupils' sentences reveal the dominance in the perception of the black colour, it leads the pupils to overcome the sequential order of the beads.

(xxv) (TR) Pupils are invited to express the situation with the *other* mathematical formulation. Alberto proposes, raising general satisfaction in the class:

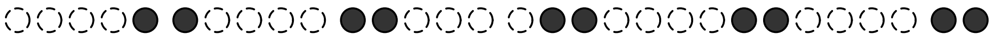
$$(2 + 4) \times 5$$

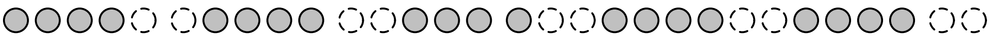
(xxvi) (IR) The class is asked to explain how the necklace was "viewed" by those who wrote $4 \times 5 + 2 \times 5$ and Alberto, who wrote $(2 + 4) \times 5$.

(xxvii) (C) Giulia: "We count how many the black beads are and then the grey ones and put them together"[15]. Giada: "Alberto adds the four grey beads to the two black beads and repeats them 5 times".

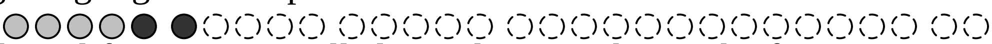
(xxviii) (IR) A 'mental experiment' is proposed to the class: "Imagine a completely dark place where you can switch on a spotlight to illuminate the things you want to highlight every time. In this dark place there is your necklace: draw a sketch showing, as under the spotlight, the necklace seen in the first case, i.e. by the class, and then another sketch showing the necklace in the second way, i.e. seen by Alberto."

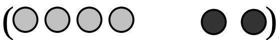
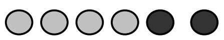
(xxix) After some uncertainty, pupils highlight the 'two moments' in which the necklace is perceived in the first case. They draw two sequences of beads in which they highlight separately the beads of different colours leaving the others white:





The necklace 'seen' by Alberto needs 'one single moment' in which the spotlight highlights the repeated module.



Stencils and friezes are recalled: pupils agree that in the first case two stencils are needed (), and in the second case one stencil is enough (). A pupil says that it is more convenient and recalls the already encountered economy principle.

(xxx) The need for a mathematical expression comes back to make Brioshi understand that the two ways of 'seeing' the necklace are *equivalent*. Pupils are quick in proposing the following expression:

$$4 \times 5 + 2 \times 5 = (2 + 4) \times 5$$

Steps xxiii – xxx

A third problem (xxiii) leads to representations referring to the only expression (A) $a \times c + b \times c$ (xxiv), although the formulation of the text seemed to induce (B): $(a+b) \times c$. The 'dominant' perception confirms what emerges from other activities of the ArAl Project, concerning the search for regularities. In front of a sequence (frieze, necklace, etc.) characterised by alternating groups of elements, for instance two, pupils identify alternation *more regularly than* repetition *of a module made of both groups. The hypothesis we formulate is that perceiving* independent *elements is more*

[15] Again Giulia uses the verb "to count" as condensing actions, operations that express "the way in which" in a compact, condensed way.

spontaneous than perceiving relationships *between elements*[16]*. Perception of the alternation hinders the identification of the* structure *of the sequence and inhibits representation (B). A* field restructuring*, in Gestaltian terms, is necessary.*
The teacher's invitation leads to the emergence of (B) (xxvi) and to a verbal description of the mental models underlying (A) and (B) (xxvii-xxviii). An 'experiment' is proposed to favour a re-reading of the context (xxix): this leads the class to elaborate on visualisations that make the two different perceptions transparent *(xxx) and to an intuition of the equality of the two representations.*

Conclusions

We now simply give a short indication about the prosecution of the didactical path. The key point is focussing pupils' attention on the comparison of the arithmetical writings arising from the solutions of faced problems, in order to lead them to grasp the general validity of the equality $(a+b)\times c = (a\times c)+(b\times c)$. The main steps of this part of the path are: a) problem situations with iconic support differing for both context and numerical values, in order to favour the two different perceptions of the field; b) problem situations similar to the previous ones, without iconic support that differ for both context and numerical values; c) problem situations proposed in two partially different versions, in order to strengthen the sense of the two representations; d) comparison among problem situations and the related expressions representing their solutions, in order to favour the understanding of the independence of equalities from numerical values and types of data; f) framing of the various equalities in a scheme and conceptualisation of the property. The detailed analysis of these steps of the path and the reflections about the ways in which the pupils conceptualize the property will be the topic of another paper.

References

Malara, N.A., 2004, From teachers education to classroom culture, *regular lecture at ICME 10*, to appear

Malara N.A., Navarra G.: 2001, "Brioshi" and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language, in Chick, E. & al. (eds), *proc. 12th ICMI Study on Algebra*, vol. 2, 412-419

Malara, N.A., Navarra G.: 2003a, *ArAl project: arithmetic pathways towards favouring pre-algebraic thinking*, Pitagora, Bologna

Malara, N.A., Navarra G.: 2003b, Influences of a procedural vision of arithmetic in algebra learning, *proc. WG 2 - CERME 3*, (Bellaria, Italy, february 2003)

Mok, I.A.C.: 1996, Progression of students'algebraic thinking in relation to the distributive properties, PhD thesis, King's College London, The University of London

Tirosh D., Hadass R., Movshovic-Hadar N., 1991, Overcoming overgeneralizations: the case of commutativity and associativity, *proc. PME 15*, vol.3, 310-315

Vermeulen, N., Olivier, A., Human, P.**:** 1996, Students' awareness of the distributive property, *proc. PME 20*, vol. 4. 379-386.

[16] Another hypothesis is that, since 'seeing' is a procedural activity, the diversity of colours breaks the perception of the unity of a module, highlighting two subsequences, and this would induce a *distributed* vision (*A*). The other one (B) is more evolved because it concerns a vision that goes beyond colours and captures the *unitary structure of the bicolour module*. The two hypotheses are being compared and analysed in depth.