THE ACHIEVEMENT OF THE EQUIVALENCE OF REPRESENTATIONS OF THE DISTRIBUTIVE LAW WITH YOUNG PUPILS

Giancarlo Navarra and Antonella Giacomin[1]

Abstract

The paper illustrates some aspects of a study about the distributive law carried out with 8-11 aged pupils within the theoretical framework of early algebra. Our hypothesis is that a path leading to understanding the law should start from an analysis of the ways in which perception leads to different mental images that influence the choice of either the (a+b)×c or the (a×c)+(b×c) representation. Understanding these dynamics is a fundamental step for both constructing a meaningful approach to the law and favouring its gradual conceptualisation.

Introduction

This work is part of the *ArAl project: Arithmetic pathways towards favouring pre-algebraic thinking*, which was designed to revisit arithmetic teaching in a pre-algebraic perspective (Malara & Navarra, 2001, 2003a, 2003b). It concerns a part of a teaching path centred on problem solving activities finalised to construct in pupils an experiential basis for an objectification of the distributive law, through collective discussions for sharing and reflection[2]. The distributive law, together with associative and commutative laws, plays a key role on both the arithmetical and algebraic side and more generally in the production of thinking via algebraic language. In usual teaching practice, however, they are rarely explored through meaningful activities. Pupils are thus led in the position not to understand their sense, and not to recognise their value on the operative level. The spreading of this phenomenon is documented by studies concerning teachers (Tirosh *et al.*, 1991) and by studies focusing on a conscious learning of arithmetical properties and of the distributive law in particular (Mok 1996; Vermeulen *et al.*, 1996). In the ArAl project the law appears in several activities and it is due to this pervasiveness that we deem important to identify a path leading to an objectification of the law itself through problem situations that highlight its genesis. Our current results stress the initial influence of perception on the construction of mental images, useful for conceptualising the property, and the effectiveness of processes of sharing.

1. Methodology

The key point of the introductory path to the law is focussing pupils' attention on the comparison of the arithmetical writings emerging from the solutions of tackled problems, in order to lead them to grasp the general validity of the equality (a+b)×c = (a×c)+(b×c). The main Steps (for pupils aged 8 to 11) are:

S1) concrete problems which lead to the conceptual embryo of the law;

[1] GREM, Department of Mathematics, University of Modena and Reggio Emilia, Italy, ginavar@tin.it, antongiac@tin.it.

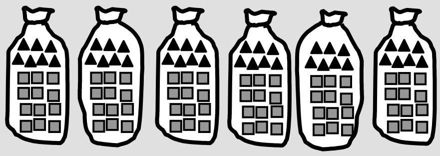
[2] The theoretical frame of the project is sketched in the quoted papers. Its English version is in www.matematica.unimo.it/0attività/Formazione/ArAl.

S2) various problem situations with iconic support, in order to favour the two different perceptions of the field;
S3) problem situations similar to the previous ones, without iconic support;
S4) problem situations proposed in two partially different versions, in order to strengthen the sense of the two representations;
S5) word problems and drawings made up staring from the two representations; the inverse process favours reflection on their structure;
S6) comparison between problem situations and the related expressions representing their solutions, in order to favour the understanding of the *independence* of equalities from numerical values and types of data;

2. S1: from perception to representation

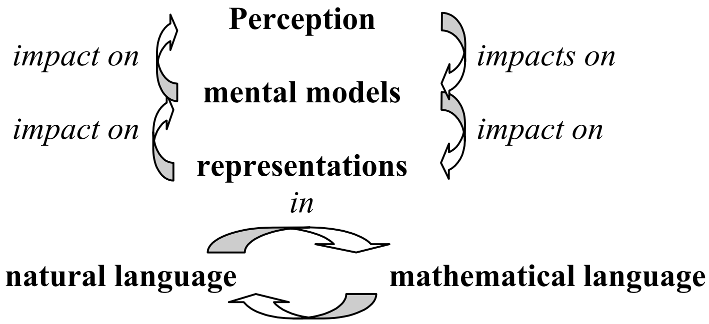
Phase S1) of the path leads pupils – through the exploration of concrete problem situations – to encounter the conceptual embryo of the distributive law. The objective is to build up premises to subsequent deeper studies of the property as *mathematical object*.
The following is the starting point of a typical problem situation:

The teacher puts 6 bags (made of a non transparent tissue) on the desk and explains that each one has the same content, made of 7 triangles e 12 squares.

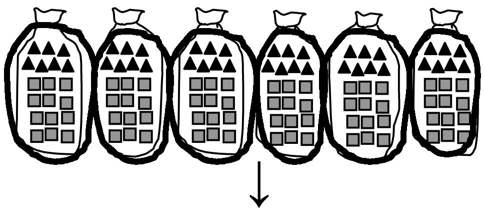


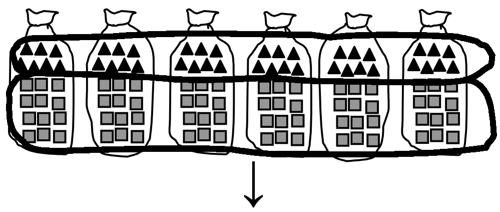
Pupils are asked to count the total amount of objects. It is strongly pointed out that we are *not interested in the number* of objects, but rather in *the reasoning followed* to find it.

During this phase a system of relationships is developed, and it can be visualised through the following model:



Each pupil *perceives* the situation depending on his/her own sensitivity and this leads to *different mental models*; in this moment we are interested in two models among these, because they lead to two different groups of representations in mathematical language for the distributive law, which can be defined as *transparent* (a_1, b_1), and *opaque* (a_2, b_2).





(a_1) $(7 + 12) \times 6$ (a_2) 19×6 (b_1) $7 \times 6 + 12 \times 6$ (b_2) $72 + 42$

Natural language is for pupils the main mediator towards an understanding of the two different *intentions:* (a) "We add the twelve squares to seven triangles and we repeat them six times"; (b) "We count how many triangles there are, and then the squares and then we put them together". The collective comparison of representations– especially the transparent ones – leads pupils to grasp readings that differ from their own, thus inducing a feedback that favours both the understanding and the appropriation of alternative mental models that in turn influence a different perception of the problem situation. Pupils are protagonists of this *field restructuring* (in a Gestaltic sense) through which a metacognitive reading of the activity is exalted.

This first phase's objective is then to *educate* pupils' perception, so as to make them aware that *different* ways of perceiving a situation exist, and that some of these may be more *productive* from a mathematical viewpoint: The purpose is to favour an initial conceptualisation of the fact that the two different representations can be reduced to the same *object*, and therefore they are equivalent. In the next sections we will analyse phases S2-S5.

3. Some general remarks about two representations

Six different classes took part in the activities described in this paper, for a total number of 111 pupils, the majority of which aged 9. We refer to problem situations concerning phases:

S2) various problem situations with iconic support (5);
S3) problem situations similar to the previous ones, without iconic support (3);
S4) problem situations proposed in two partially different versions (2).

For each problem pupils were free to develop their own representations. Among the representations produced, 307– variously correct, complete, transparent– can be drawn back to the two representations, with these percentages: 55% to $(a+b)\times c$, 45% to $a\times c+b\times c$. It seems that one does not prevail on the other significantly, although the first one is more frequent.

We may think that the *unitary entity* described in the problem (the bag in the example) is *physically* stronger, although it has a heterogeneous content, than the abstract operation of counting objects following a principle of *transversal homogeneity* (see (b) in fig. 2). The pupil spontaneously *sees* the bag's content (*concrete* operation) and multiplies it (*abstract* operation) by the number of bags. The consequent representation, $(a+b)\times c$, reproduces the different complexity of the two operations at a mathematical level: f*irst* you add (a *simpler* operation), *then* you multiply (a *more difficult* operation).

In (b), instead, the two types of objects (triangles and squares) are counted separately by extracting them mentally from their containers (*abstract* operation), and then put together (*concrete* operation). Also in this case the related representation $a\times c+b\times c$ reproduces the different complexity of the operations at a mathematical level: *first* you multiply twice (*more difficult* operations), *then* you add up (*simpler* operation).
Two last remarks from a mathematical point of view: in $(a+b)\times c$ operations are put in their execution order, thus allowing a procedural reading, whereas this is not the case for $a\times c+b\times c$. However there are situations in which the latter representation is more frequent than the former, and we will briefly analyse them in the next section.

4. Representations of the law and search for regularities

In some types of problems involving objects such as necklaces, friezes, decorations, drawings made with matches, objects to be analysed are variously long sequences consisting of a repeated module, as in the following example:

Pupils are shown a belt made of rounded knobs and squared studs:

The task is the usual 'ArAl' one: to explain in either natural or mathematical language (or both) the *way* to find how many studs the belt is made of.

In these cases we may notice a 'dominant' perception that leads to the $a\times c+b\times c$ representation and confirms what emerges from the ArAl project about the search for regularities: pupils, facing a sequence characterised by alternating groups of elements, identify *alternation* more frequently than repetition of a *module*. In other words, most pupils perceive two independent sequences in this way:

 *4 × 5*

 *2 × 5*

and get the consequent expression:

$$(A) \quad 4 \times 5 + 2 \times 5$$

Only a minority 'see' the module:

and consistently construct the expression:

$$(B) \quad (4 + 2) \times 5$$

One hypothesis is that perceiving different elements, *independent from each other* (circles, squares), is more spontaneous than perceiving relationships among elements (circles together with squares). Another hypothesis is that, since seeing is *procedural,* the diversity of shapes may *break the perception of the module's unity*, highlighting the two sub-sequences: this would induce the *dis-*

tributed vision *(A)*. (B) is more evolved because it concerns seeing beyond the shape and capturing the *unitary structure of the module*.
The perception of alternation thus hinders the identification of the *sequence structure* and inhibits representation (B) slowing down the path towards the a-chievement of a 'general law' (as in the example).
A *field restructuring* (in gestaltic terms) is needed.

5. Phase S4: Problems in two versions

During phase S1 pupils are guided towards grasping the equivalence between the two representations; in S2 and S3, through a collective comparison of the proposed representations they reinforce their understanding of the equivalence.
In S4 we constrain the situation by proposing word problems in two versions, built as to suggest $(a+b)\times c$ in the first one and $a\times c+b\times c$ in the second one. The didactical hypothesis is that, through discussion and *a posteriori* comparison between texts and representations produced by pupils, they can consolidate the concept that beyond textual differences, there is a single situation and therefore representations of calculation modalities are equivalent.
Unexpectedly, this phase, developed with 92 pupils aged 9 and from three different classes gives surprising results, and allows us to face the delicate aspect of the possible *variants* of the two representations.
Let us illustrate this through an example: the 'Fish problem':

First version	Second version
A big park is full of games. At the four corners there are four equal tanks with fish. In each tank there are 5 red and 2 silver fish. Giuseppina likes these fish very much and she always goes there to see them. She tried to count them but they always escape and move all the time. Explain to her how she can find the number of fish in the four tanks in the park. *Expected representation: (5 + 2) × 4*	A big park is full of games. At the four corners there are four equal tanks with fish. On Tuesday the dealer came and put 5 red fish in each tank. But in this way the tanks were a bit empty and therefore he came back on Thursday and put 2 silver fish in each tank. Explain how you can find the number of fish swimming in the park's tanks. *Expected representation: 5 × 4 + 2 × 4*

The unexpected result is that the 34 protocols show that the two versions do not actually influence pupils significantly: not only representations are of both types, but they are often inverted with respect to our expectations, with a prevailing number of writings that trace back to $(5+2)\times 4$. This suggests the hypothesis that the context– *independently on how it is structured*– is perceived differently by different individuals, who elaborate a mental model to be later translated into mathematical language. Therefore it becomes even more important to compare different perceptions, so as to favour the *field restructuring* as well as the awareness that among *different* ways of perceiving a situation some turn out to be more *productive* from a mathematical viewpoint– sometimes in decisive ways- to identify the *structure* of a sequence.

6. From the discovery of the equivalence towards generalisation

The comparison of different problem situations enables pupils to grasp the persistence of the equality of representative expressions of the two counting strategies although data vary in value and typology. The route to generalisation is still full of obstacles though.

One initial obstacle, at this age level, is represented by interferences between *procedural* aspect and *relational* aspect of a representation. Let us explain this with an example related to the Fish problem. Representations (at various levels of correctness, anyway similar to the two conventional ones) globally produced by pupils are distributed as follows:

First version		Second version	
Expected representation: (5 + 2) × 4		*Expected representation: 5 × 4 + 2 × 4*	
Ppils' representations		*Pupils' representations*	
(5 + 2) × 4	*[10]*	*(5 + 2) × 4*	*[7]*
5 × 4 + 2 × 4	*[6]*	*5 × 4 + 2 × 4*	*[6]*

From discussions about representations deeply different attitudes emerge, ranging from naive ones: "In the solution 7 times 4[3] I wonder how Giuseppina knows that there are seven fish if they move continuously. How did she count them?" to more aware ones, that identify the commutative law on the basis of an exchange between numbers. But it is from the comparison of paraphrases of representations that an extremely important aspect emerges, from the point of view of the *construction of meanings*. Let us see which one.

In one of the classes, after screening wrong representations through discussion, the definitive discussion is set on the basis of the remaining ones, leading to the choice of representations to be sent to Brioshi[4]:

First version		Second version	
Expected representation: (5 + 2) × 4		*Expected representation: 5 × 4 + 2 × 4*	
Pupils' representations		*Pupils' representations*	
(a_1) *(5 + 2) × 4*	*[4]*	(b_1) *4 × (5 + 2)*	*[7]*
(a_2) *5 × 4 + 2 × 4*	*[2]*	(b_2) *(4 × 5) + (4 × 2)*	*[6]*

Many remarks capture both the correctness of paraphrases and their equivalence ("They seem to be equal to me", " (a_1) is similar to (a_2) and (b_1) to (b_2)", etc) but it

[3] The pupil refers to one of the many *opaque* representations produced in classes, in this case 7 × 4 instead of the more *transparent* (5 + 2) × 4.

[4] Brioshi is an imaginary Japanese pupil (variably aged according to the age of his interlocutors) and is a powerful support within the ArAl project. He was introduced to make pupils aged between 7 and 14 approach formal coding and a difficult related concept: the need to respect rules in the use of language, need which is even stronger when engaging with a formalised language, because of the extreme synthetic nature of the symbols used in them. Brioshi is able to communicate only through a correct use of mathematical language and enjoys exchanging problems and solutions with foreign classes, through a wide range of instruments, such as messages written on paper sheets or exchanges through the Internet.

is clear that they do not yet refer to the mathematical objects, but rather to a frame of semantic references to the problem situation. In actual fact many define writings (a_1) and (a_2) as clearer than the others because they better express situations as they are presented: the number of red fish (5) precedes that of silver fish (2) and anyway animals have a stronger ‘emotional impact’ than tanks (4). The order of data and operations reflects the *temporality* characterising the description. Moreover, teachers make pupils used to link the datum having the same features of the product (the number of fish) to the multplicand. This semantic link to the text makes writings (b) poorer, because they *invert the order of data* and thus seem to be unfaithful to the text.

This point of view should emerge and become object of a debate and deep discussion with the class, because it leads to an increased control of both semantic and syntactic aspects. For instance, in tasks like ‘Add up 8 and 3’, the teacher should point out the fact that the commutative law holds, but at the same time should provoke reflection on the fact that the translation *8+3* does not express the ‘literal’ sense of the process (‘*first* there is 3, and *then* you add 8’), but it turns it into an equivalent one, induced by the order in which the two numbers are given in the task. The most *faithful* translation would actually be *3+8*.

We believe that a constant comparison of these aspects promotes a better understanding of both and, at the same time, underlines their differences, thus favouring the necessary *detachment* from the initial situation and a getting closer to generalisation.

Similar opportunities for a deep study are provided by S5 activities in which pupils are asked to express sentences in mathematical language through drawings.

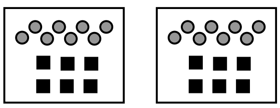
7. Interpreting a message by Brioshi through drawings

Pupils are asked to represent a message by Brioshi through a drawing; half class must represent the left side and half the right side.

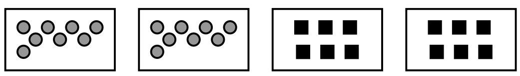
$$(8 + 6) \times 2 = 8 \times 2 + 6 \times 2$$

We expect this type of drawings:

(a) $(8 + 6) \times 2$

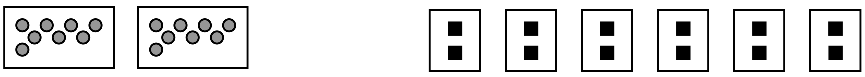


(b) $8 \times 2 + 6 \times 2$

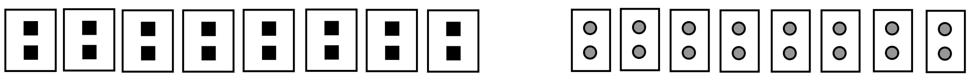


The number of correct representations of situation (a) is twice the others. Some of (b)s make pupils puzzled:

(b_1)



(b_2)



Discussion leads to the respective translations:

(b_1) $8 \times 2 + 2 \times 6$ (b_2) $2 \times 8 + 2 \times 6$

Also this situation turns out to be very productive because what the class perceives as *unfaithfulness* of these drawings to the two representations favours a reflection on their *sense,* and therefore a detachment from the concrete situation.
Very often authors themselves find it difficult to express the reasons of their choices clearly, being influenced in some cases by perceptual conditioning (as for instance in (b_2) 'seeing' the '2' since it is repeated twice in the formula, whereas '8' and '6' appear only once). Despite this, trying to interpret such different drawings, pupils *go deeply* into the equivalence, reinforcing the concept that it links various writings, beyond their more or less consistency with the context.

8. (Partial) conclusions

The achievement of the meaning of the distributive law represents the outcome of a process requiring long time and knowledge of modalities through which pupils can actually get to it. This aspect entails an engagement in both research and teacher training.
The perceptual aspect seems to prevail in the first phases since it provokes the shaping of mental models underlying the mathematical representations of the law. The semantic link with the context is strong and extremely various activities are necessary to make pupils get detached and become aware of a general validity of representations, independent from specific numerical values.
Collective sharing and discussion on representations elaborated by pupils (in mathematical, natural, iconic language) play a fundamental role in this sense; they favour a detachment from problem situations and their framing in a single scheme independent from the numerical values of the three involved data, from the type of data and from the counting strategy adopted; in other words, the transition to *generalisation*.

References

Malara, N.A. (2004). From teachers education to classroom culture, *regular lecture at ICME 10*, to appear.

Malara N.A., Navarra G. (2001) "Brioshi" and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language, in Chick, E. & al. (eds), *proc. 12th ICMI Study on Algebra*, vol. 2, 412-419.

Malara, N.A., Navarra G. (2003a). *ArAl project: arithmetic pathways towards favouring pre-algebraic thinking*, Pitagora Editrice, Bologna.

Malara, N.A., Navarra G. (2003b). Influences of a procedural vision of arithmetic in algebra learning, *proc. WG 2 - CERME 3*, (Bellaria, Italy, February 2003).

Malara N.A., Navarra G. (2005). Approaching the distributive law with young pupils, *Proc. of WG 3 - CERME 4,* (Sant Feliu de Guìxols, Spain, February 2005).

Mok, I.A.C. (1996). Progression of students' algebraic thinking in relation to the distributive properties, *PhD thesis,* King's College London, The University of London.

Tirosh D., Hadass R., Movshovic-Hadar N. (1991). Overcoming overgeneralizations: the case of commutativity and associativity, *proc. PME 15*, vol.3, 310-315.

Vermeulen, N., Olivier, A., Human, P. (1996). Students' awareness of the distributive property, *proc. PME 20*, vol. 4. 379-386.