Early Algebra: Theoretical Issues and Educational Strategies for Bringing the Teachers to Promote a Linguistic and Metacognitive approach to it

Annalisa Cusi[1] , Nicolina A. Malara[1], Giancarlo Navarra[1]

[1] Mathematics Department, University of Modena and Reggio Emilia, Italy

Abstract. (i) After a brief overview of the studies which brought to the rise of early algebra, (ii) the issues of socio-constructive teaching as well as of the more complex role of the teacher are discussed and the importance of reflecting upon teaching and learning processes for the professional development of teachers is underlined. (iii) Our view on early algebra and our studies about it are then introduced, highlighting the role played by metacognition in both pupils' conquest of the mathematical meanings that emerge from the classroom- based action and teachers' development of a capacity of having control of and predicting the effects of their own actions. (iv) We dwell on both instruments and modalities we used to lead teachers to reflect upon the impact of their own behaviours on pupils' mathematical constructions, and (v), in this concern, we underline the high value of analytical comments made by teachers in written form on the transcripts of classroom- based processes.

Keywords: commented teacher's transcripts, critical reflections, early algebra, languages, metacognitive attitudes, teachers educational strategies, teachers' role, socio-constructive teaching

1 Introduction

The idea of giving space to early algebra at K-8 school level in association with a socio-constructive practice of teaching seems to be increasingly spreading. This does not mean that syntactic activities typical of secondary school should be anticipated to lower school levels, but rather that in elementary number theory more room should be given to activities concerning numbers in relational

terms, so that pupils might be led to compare representations and equal different representations of the same mathematical object, to detect analogies, to generalize and identify properties. In other words, it would be appropriate to *revise arithmetic in a pre-algebraic perspective*, reducing a typical algorithmic treatment and setting the ground for the development of algebraic thinking. The aim is to get students to construct, starting from their early school years, a set of experiences that make the study of algebra in its formal aspects, meaningful and justified. In this way, the approach to algebra should be facilitated and the typical and widespread difficulties students meet when they access higher secondary school, minimized. At the same time, they should be made aware of the potential of algebraic language as a tool for thinking.
Several countries have nowadays included this theme, more or less explicitly, in their National Curricula, though this was done in the framework of their specific cultural and educational features, and recently, comparative studies about the different approaches and methods have been carried out (see for instance Kieran 2004, Cai & Alii 2005). Moreover, the introduction of early algebra in the different educational policies has been promoted by the indication of the British Department for Education (DFE 1995) and, even more, by those of the National Council of Teachers of Mathematics (NTCM 1998, 2000), together with the exploratory studies of didactical implementations carried out at the level of research (we will deal with these studies later in the text).

1.1 In Europe

In Europe, the initial, embryonic, teaching proposals in the spirit of early algebra date back to the 70s, as a follow-up on curricula of two different and somehow opposite tendencies: the psycho-pedagogical trends which underline the importance of experience and discovery in learning, and the structuralist trend that suggested an algebrization of the mathematical teaching contents.
In those years, the naïve set theory comes to the front in teaching and the concept of binary operation, with its properties, becomes the fundamental basis of the arithmetic-algebraic area. On the one hand, this approach opens the way to a characterization of the structures of

the different number sets; on the other hand, it makes both the concept of relation and the modeling processes central, due to the fact that they lead to embed the concept of function in the objects of algebra.

With this approach, starting from primary school, in elementary number theory importance is given to the relational aspects of numbers, to the symmetry of the equality, to the recognition of equivalent representations of numbers, to the valuing of arithmetic properties for ordering numbers. At the same time, room is given to the study of relationships in realistic contexts and with reference to different number sets, with the joint detection of variable data for pairs of quantities[1].

The importance attached to modeling processes leads to a review of the teaching of algebra - up to that moment mainly viewed in purely syntactic terms - as well as to pose a higher attention to algebraic language as a representational tool. Pioneer studies carried out by English scholars offer teaching experiences aiming to generalization by means of realistic situations in several contexts, but also in situations within mathematics, often playful and even referred to proof (see, for instance Bell 1976, Bell & Alii 1985, Harper 1987).

1.2 From traditional algebra to early algebra

In the actual practice, the new views on teaching and learning come to conflict with the view of traditional algebra. For this reason, but not only for that, diagnostic studies on pupils' difficulties are carried out, also taking into account issues related to both modeling and interpretation of formal expressions. Classical studies in this respect are those of Both (1984), Kucheman (1981) e Kieran (1989, 1992), Lee & Weeler (1989), which point out that many difficulties and blocks in the learning of algebra descend from a teaching of arithmetic essentially centered on the aspects of calculation and very little focusing on its relational and structural aspects.

[1] In the early '70 a noticeable project in Europe was the Hungarian project for primary school directed by T. Varga, which envisaged activities of this type since the first two years of primary school. A simplified version of the project, named Ricme Project, was implemented in Italian by the National Research Council.

During the ICME 6 Congress (Adelaide 1984), this topic is already debated and a proposal is made to introduce relational, generalization-type and modeling activities in primary school. However, an important step toward the constitution of early algebra as a disciplinary area is made at the ICME 7 (Quebec 1992); in that congress a proposal is made for objectifying a new space of arithmetic teaching, the *pre-algebra*, aimed to the development of 'pre-concepts' useful to algebra, i.e. advanced arithmetic concepts, of a structural type, setting an experiential and conceptual basis for the connection with more abstract and formal algebraic concepts (Linchevski 1995).
In those years, several scholars point out the importance for pupils to acquire the 'sense for symbols' (Arcavi 1994) through a variety of activities which may help them develop abilities, understanding and ways of feeling that may eventually lead them to act in a flexible and instinctive way within a system of symbols, to move around in wider or different systems of symbols and to co-ordinate interpretations of formulae in various solution worlds (Arzarello 1991, Arzarello & Alii 1993, Gray e Tall 1993, Filloy 1990, 1991, Kaput 1991, Lins 1990). In US, debates about the algebrization of the K-12 curriculum from kindergarten to secondary school are undertaken (Kaput 1995).
At the ICME 8 Congress held in Sevilla (1996) Kieran characterizes elementary algebra through three types of activities[2] at increasing complexity levels, setting at the first level generational activities, i.e. those through which the objects of algebra can be constructed linking meanings to experience. The second half of the '90s is characterized by a high number of experiments on generational algebraic activities mainly addressing 11-13 years old pupils. Some studies theorize socio-constructive models of conceptual development in algebra, in which the influence of the classroom environment on learning, as well as the importance of the role of the teacher are emphasized, in the framework of a view of algebra as language (see for instance Da Rocha Falcão 1995; Meira 1996; Radford 2000).

[2] 1) Generational activities; 2) Transformational activities; 3) Global, at meta-level activities.

Starting from year 2000, the issues of early algebra become of increasing interest in the International community, as it is shown by studies about early algebra at the 12th ICMI Study '*The future of the teaching and learning of algebra'* (Cick & Alii 2001), and other collective studies, such as the forum on Early Algebra at the PME 25 (Ainley & Al. 2001), the Special Issue on Early Algebra of the ZDM Journal (Cai & Alii 2005), the international seminar "*Pathways to Algebra* " organized by D. Carraher in France (Evron, June 2008).
All these studies mainly concern issues of implementation of innovative activities in primary school and analyze pupils' behaviours and learning. Several studies also deal with the problem of a suitable teacher training with focused interventions on their professional development (see for instance Carpenter & Franke, 2001, Carpenther et Alii 2003, Dougherty 2001, Blanton & Kaput 2001, 2002, Kaput & Blanton 2001, Menzel 2001).
Our studies are in the line of this last trend and develop within the *ArAl Project*: *teaching sequences in arithmetic to favour pre-algebraic thinking* (Malara & Navarra 2003). The Project started in 1998 on the basis of our previous studies (Malara & Iaderosa 1998) at secondary school level (grades 6-8) and it is designed for primary school in the perspective of a continuity between the two school levels. In this project we claim that *the main cognitive obstacles to the learning of algebra arise in unsuspected ways in arithmetic contexts and may impact on the development of mathematical thinking*, mostly due to the fact that many students only have a weak conceptual control over the *meanings* of algebraic *objects* and *processes*. Our aim is to make teachers aware and caring about this situation and provide them with instruments that enable them to design and implement powerful interventions to face it.
Before getting into the issues we dealt with, we briefly discuss the more general issue of the role of the teacher in socio-constructive teaching.

2. Socio-constructive teaching and teacher training

In the teaching of mathematics the socio-constructive model of teaching is spreading, since it is viewed as suitable to educate students (mainly aged between 6 and 14) to work collectively as well as to favour their acquisition of flexibility in thinking. According to this model, teachers should start their action from the devolution to students of purposefully designed problem situations that may bring about the emergence of particular mathematical concepts and properties. The core of the model is the view of students as makers of their own knowledge: it develops through argumentation and exchanges of ideas, up to the collective systematization of the results obtained and a reflection upon meanings and role of those results.
The teacher plays a number of roles in this model: he designs teaching sequences suitable to promote mathematical constructions by the pupils, creates an environment that favours both mathematical exploration and argumentation, uses communicative strategies that favour both interaction and sharing of ideas. Moreover, depending on the mathematical questions at stake, he predicts the students' possible thinking processes, makes hypotheses about the possible answers to some key questions and, most of all, faces the possible evolution of the mathematical discourse along lines that may even significantly diverge from the predicted one.

2.1 The mathematical discussion

The whole-class mathematical discussion plays a central role in the model. In order to be able to fulfill the task, the teacher should master notions and abilities that go beyond the mere knowledge of the discipline:

- from a social viewpoint, *to be able to create a good interactional context*, by stimulating and guiding the argumentative processes (mediating argumentation in the words of Schwarz et Al., 2004) easing communication, listening, evaluation and capacity of producing a counter-argumentation (Wood 1999);

- *to activate socio-mathematical norms* that lead to check the acceptability of a solution, to evaluate different solutions, to appreciate the quality of a solution (Yakel & Cobb 1996);
- *to determine the direction of the discussion* in its various phases, filtering students' ideas, so that their attention may be focused on the contents the teacher views as more relevant and meaningful (Gamoran Sherin, 2002);
- *to harmonically enact modalities* (Anghileri, 2006) such as: reviewing (focusing pupils' attention on aspects of the activity that may favour the understanding of the underlying mathematical ideas); restructuring (encouraging students to reflect upon and clarify to themselves what they have understood, in order to favour both development and strengthening of mathematical meanings); re-phrasing of students' utterances (re-formulation of what one or more pupils claimed to highlight and clarify the argumentative processes developed in the classroom); using probing questions (posing questions in order to investigate on students' statements, with the aim of leading them to clarify what they said and favour a development of their thoughts);
- *to involve pupils in metacognitive acts* (transactive utterances in the terminology of Blanton, Stylianou and David, 2003), to enable them to internalize collective argumentative processes.

2.2 The role of the teacher's reflection

Several researchers underline the value of the teacher's critical reflections on the classroom-based processes (Mason 1998, Jaworski 1998, Shoenfeld 1998) and most of all, of the practice of sharing these reflections (Borasi et al. 1999, Da Ponte 2004, Jaworski 2002, Malara 2003, Malara & Zan 2002, 2008, Potari & Jaworski 2003) for the acquisition of the above described competences.
In particular, Mason (2002), and we will get back to this, to make teachers acquire the capacity of carefully observe themselves in the class-based action, suggests the constant practice of the '*discipline of noticing*', recommending that reflections be shared among colleagues in order to be validated. Jaworski (2004) points out the efficacy of "communities of enquiry" (mixed groups made by

teachers and researchers) highlighting the fact that participating in these groups brings about a taking on of identity by the teacher.

On this basis, the hypothesis which is outlined requires a change of perspective by the teacher. He should re-learn to manage *socio-cognitive* processes (experience in the class), drawing on the theoretical frameworks proposed to him, comparing them to his own epistemology, thus being fruitfully and significantly enriched in both his culture and work in the classroom. In this way, the teacher may avoid the feeling of powerlessness in front of paradigms that are too abstract or self-referential to become reliable keys for reading his own experience, or rather paradigms for an intervention on his own practice.

This hypothesis holds for trainee teachers as well, because it is about methodological aspects, even before mathematical ones: a background in mathematics is a necessary condition, but not a sufficient one, to become a good mathematics teacher.

Mason (2002) starts up his text on the discipline of noticing with the following maxim:

> 'I cannot change others; I can work at changing myself'.

In many respects, the latter strictly links to our discourse. As a matter of fact, many teachers, especially the elderly ones, believe that they can intervene and change their pupils without having tried to consciously change themselves. In other words: *without putting themselves critically in front of their own practice, investigating it.* Mason also writes:

> "Working to develop your own practices can be transformed into a systematic and methodologically sound process of 'researching from the inside', that is, of researching yourself".

Hence it is not a matter of assuming didactical key ideas and use them as 'methodological navigators', it is rather a matter of starting up a continuous reflection upon oneself as a professional in education, making use of theoretical supports that may bring about the awareness that a continuous *transformation* is needed. The final aim is to get to go beyond the idea that some little adjustments (such as changing the textbook, using new technologies, attending some training courses) can be enough to produce effective changes in pupils' learning.

But, in order to be effective, transformation requires an essential condition: that one trains himself to understand *in which directions* transformation should be promoted. A *fruitful exchange between theory and practice* may bring the teacher to develop capacities at two levels: at one first level, to *grasp signs* in all that contributes to define his own condition, both *in the field* – in his activity in the classroom – and in the construction of his own theoretical instruments, through readings, congresses, training courses; at a second level, to *elaborate on the grasped signs* so that they become part of his rooted cultural background.
The development of a capacity of grasping signs is achieved only through the teacher's increasing awareness in learning to transform thousands of occasionally noticed things into a tool of one's individual methodology, deriving from a relation between the capacity of noticing, the motivation to intervene and the acquisition of instruments that suggest *how* to intervene.
Concerning the capacity of noticing, Mason adds:

> 'Every practitioner, in whatever domain they work, wants to be awake to possibilities, to be sensitive to the situation and to respond appropriately. What is considered appropriate depends on what is valued, which in turn affects what is noticed. … noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next. It is almost too obvious even to say that what you do not notice, you cannot act upon; you cannot choose the act if you do not notice an opportunity.'

Hence the question is: *What* should the teacher *notice*? *Who* would teach him/her to notice this *that?*
What we maintain is: *the teacher is firstly a mentor of himself,* through a continuous engagement, with the aware support of a reference map as well as of suitable stimulating tools (continuously and critically reviewed, as we will see in the next paragraphs), which allow him to start up an exploration of a cultural baggage which is certainly familiar to him, but at the same time, needs to be re-considered from different viewpoints, through a process that will gradually lead him to a *forma mentis* that is profoundly different to that of the previous stage. In our case, the point of arrival will be a re-reading of one's conceptions with relation to arithmetic and algebra. *The teacher must be active protagonist of his own development* in his approach to early algebra.

In this respect, the main tool our project (ArAl) refers to, is a set of theoretical constructs, partly drawn from other constructs and partly original, organized in a Glossary, which we will present here, thus deepening the description of our conception of early algebra.

3. Early algebra as a meta-subject

The linguist S. Ferreri (2006) writes:

> 'You get to the keywords of a discipline through a slow work of foundation of the basic concepts of the respective disciplinary areas. Appropriating the meaning of words, of some meaningful words, is a way to stabilize, conceptualize and master a specific knowledge domain, as its contents might otherwise remain not grasped. In fact, the word is viewed as a permanent trace of a construction of knowledge, joint in memory to other pre-existing words; as a capacity of making explicit a stage of the process of knowledge which is shaping up. Words that show one's degree of control over knowledge. Words, as portions of knowledge that can represent itself'.

From our perspective, the set of keywords of early algebra does not refer to a single discipline (either arithmetic or algebra) and to its terms. It defines its limits starting from both disciplines but ends up assuming a different identity, mainly original. We might talk about a meta-subject which objects are not objects, processes, properties of the two subjects, but rather the genesis of a unifying language. A meta-language, as such.

Early algebra arises from a need – to favour the construction of meanings in very young pupils since their first approach to arithmetic - and is based on some principles in our view. Let us list some of the most important ones:

- The *anticipation of generational pre-algebraic activities* at the beginning of primary school, and even before that, at kindergarten, to favour the genesis of the algebraic language, viewed as a generalizing language, since the pupil is guided to reflect upon natural language; it is from the analogy between the modalities of development of the two languages that the theoretical construct of *algebraic babbling* comes out (we will get back to this in section 5.1).

- The *social construction of knowledge*, i.e. the shared construction of new meanings, negotiated on the basis of the shared cultural instruments available at the moment to both pupils and teacher. Arithmetic and algebraic knowings are both central, but they need to emerge and strengthen themselves through the coordinated set of individual competencies, which are the main resource on which they are constructed.
- The *central role of natural language* as main didactical mediator for the slow construction of syntactic and semantic aspects of algebraic language. Verbalization, argumentation, discussion, exchange, favour both understanding and critical review of ideas. At the same time, through the enactment of processes of translation, natural language sets up the bases for both producing and interpreting representations written in algebraic language. From this centre, attention is then extended to the plurality of languages used by mathematics (iconic, graphical, arrow-like, set-theory language, and so on).
- *Identifying and making explicit algebraic thinking, often 'hidden' in concepts and representations in arithmetic.* The genesis of the generalizing language can be located at this 'unveiling', when the pupil starts to describe a sentence like $4\times2+1=9$ no longer (not only) as the result of a procedural reading 'I multiply 4 times 2, add 1 and get 9', but rather as result of a relational reading such as 'The sum between the product of 4 times 2 and 1 equals 9'; i.e. when he/she talks about mathematical language through natural language and does not focus on numbers, but rather on relations, that is on the *structure* of the sentence.

The teacher plays a central role in these processes but, in order to be able to do that, needs to understand how, and most of all why, the construction of mathematical concepts needs to be supported by a setting made of solid linguistic and methodological bases, but also social and psychological ones. These are his/her main difficulties in the approach to early algebra. Besides reflecting upon his/her own mathematical knoweldge – and the arithmetic and algebraic ones in particular – as well as upon the consequent convictions, which inevitably affect his/her teaching, the teacher must be able to construct new *meta*- competencies and empower his/her sensitivity

in grasping the deep mutual relations between the two subjects, and the embryos of algebraic thinking underlying arithmetic concepts and representations.
Finally, getting back to Ferreri (2006), in order to 'stabilize, conceptualize and master' the meta-disciplinary knowledge of early algebra, thus avoiding that 'its contents may be not mastered', the teacher must 'appropriate the meaning of some meaningful words, portions of a knowledge that can represent itself'. These are the theoretical reasons underlying the central role of the Glossary in our conception of early algebra.

4. The role of the ArAl Glossary in teacher training

The ArAl Glossary, including the 'main' Glossary and that for kindergarten, and made at present of a hundred terms, is a reference system that allows the teacher to gradually get to an overall view of early algebra, which merges theory and practice, by approaching a linguistic view of algebra, within which a convincing control over its meanings can be constructed together with the pupils.
The structure of the Glossary is outlined so that pre-defined approaches to the included terms are not sketched. The teacher finds a plurality of routes to be explored autonomously, depending on the modalities of his approach to early algebra, his background, the age of his pupils, the themes he wants to deal with, his curiosity and so on. Each term of the Glossary is a self-sufficient entity, so to speak. The text which describes it includes some other key terms, the set of which constitutes a more or less wide Net.
A term of the Glossary may avail of a very numerous Net, but be quoted in few Nets. Vice versa, another term might have a Net with few links, but be present in many other Nets. Each term, therefore, depending on the numerousness of its Net, as well as on that of its occurrences, locates the teacher within a double process of conceptual deepening and extension: deepening of the term through the relations among the key-terms which appear in its definition; extension, since each of them is a potential stimulus to read its definition.
The terms of the Glossary may be grouped within five areas:

- GENERAL: didactical mediator, Opaque/transparent (referred to meaning), Relational thinking, Process/Product, Representing/Solving, …
- LINGUISTIC: Arguing, Algebraic babbling, Language, Letter, Metaphor, Paraphrase, Semantic/Syntax, Translating. …
- MATHEMATICAL: Formal coding, Additive form, Canonical/Non canonical form, Multiplicative form, Mathematical Phrase, Function, Unknown, Pseudo equation, Relation, Equal sign, Structure, Variable, …
- SOCIO-DIDACTICAL: Sharing, Collective exchange, Didactical contract, Discussion, Social mediation, Negotiation, …
- PSYCHOLOGICAL: Affective-emotional interference, Perception, …

The teacher must learn to promote and manage the five areas, becoming aware that:
(a) the construction of knowledge takes place through the promotion of *social behaviours* that favour *exchange* and *verbalization* in the classroom;
(b) the identification of suitable *didactical mediators* is crucial to a stable acquisition of *meanings*;
(c) it is necessary to promote activities that emphasize *metacognitive* and *metalinguistic* aspects.
Some terms of the Glossary, that the teacher meets anyway, independently on the organization of his exploration, have more numerous Nets and Occurrences than others. This attaches to them a *status of strong representativeness in the definition of early algebra from our point of view* and enables us to compose a sort of manifesto through them (in bold case in the text):

> The theoretical framework of *early algebra* supports the hypothesis that students' weak control over the meanings of algebra derives from their ways of constructing arithmetic knowledge since the early years of primary school.
>
> Algebra should be taught as a new **language,** one gets to master – through a set of shared **social** practices (**collective discussion**, **verbalization**, **argumentation**) – with modalities that are analogous to those of natural language learning: starting from its meanings (**semantic** aspects) and setting them gradually in their **syntactic** structure (a process we called **algebraic babbling**).
>
> Crucial elements in this respect are **metaphors**, didactical **mediators** in the achievement of meanings, during the conceptual progression towards generalization and modeling.
>
> In this view, natural language becomes the most important mediator in the student's experience and his main instrument of **representation** through which he can illustrate the system of **relations** (**additive** and **multiplicative** ones at the beginning) among elements in a problem situation, shifting focus from the

product to the **process,** and inducing a **translation** of the process itself into a mathematical **sentence**.

In this way, attention is shifted from the *arithmetic* objective of **solving**, to the *algebraic* one of **representing**. At the same time, mediators favour the achievement of the use of **letters**, seen as **unknown** - easier to be achieved - of **indeterminate** and of **variable**.

For a specific discussion on the structure and potentialities of the Glossary, see Malara & Navarra (2009); we are now going to illustrate some key elements of our theoretical framework.

5. Theoretical key elements in our approach to early algebra

It should be clear at this point that our perspective in the approach to early algebra is a linguistic and metacognitive one, and is based on the hypothesis that there is a strong analogy between modalities of learning natural language and algebraic language. In order to explain this point of view, we make use of the metaphor of *algebraic babbling.*

5.1 Algebraic babbling

Pre-requisite to acquire control over the syntactic aspects of a new language is a slow and *in-depth* acquisition of a *semantic* control. As we know, a child, while learning natural language, gradually appropriates its *meanings* and rules, and progressively develops them through adjustments and imitations, up to the deeper knowledge he gets to in his school years, when he learns to read and reflect upon *grammar* and *syntax*. In the same way, mental models typical of algebraic thinking should be constructed from the early years of primary school, progressively building up in pupils algebraic thinking as both tool and object of thinking, in a strict *intertwining with arithmetic,* starting from the meanings of the latter. For this reason, it is necessary to build up an environment able to stimulate informally the autonomous elaboration of formal coding for sentences in natural language, discussing them with the whole class and gradually producing a playful, experimental and continuously re-defined appropriation of the new language. The rules of this language are then located into a didactical contract, which tolerates initial moments of syntactic 'promiscuousness'. This process of construction/interpretation/refinement of 'draft' formulas is what we call 'algebraic babbling'.

An example. A fourth grade class (9 years-old) is exploring problem situations asking to *identify the relations existing between two quantities*. In one of these situations, involving variable quantities of two different types of biscuits, i.e. sponge biscuits and chocolate cookies, as usual pupils represent the relations between the two quantities in *natural language,* getting to the following correspondence rule, after selecting different formulations: 'the number of sponge biscuits is 1 more than twice the number of chocolate cookies'. The next step is to translate the sentence *in algebraic language*. Individual work leads to the following proposals:

(a) 1×2; (b) a+1×2 (a = n. of sponge biscuits); (c) sv+1×2;
(d) a×2+1; (e) sv+1×2=a; (f) sv=st+1×2;
(g) a=b×2+1; (h) a×2+1=b (a = n. of choc. cookies); (i) (a–1)×2

In the middle of the activity, the teacher will need to interpret each expression, capturing each pupil's idea, to orchestrate a discussion with pupils on the interpretation of their expressions, on their correctness, on the possible equivalencies, on the identification of the most suitable translations.

In the following, we will discuss in-depth some key concepts in our perspective: *solving* and *representing*, *canonical* and *non canonical* form of a natural number, the *equality* sign, respect of the rules in the approach to the *algebraic code*.

5.2 Algebra as a language: representing vs solving

A widespread belief in pupils is that solving a problem means identifying the result. This implies that their attention is focused on the *operations*. They should rather learn not to worry too much about the result, and therefore about the search for the operations that lead to it, and move from the cognitive to the metacognitive level, where the solver *interprets the structure of the problem and* *represents it through algebraic language*. Algebra thus becomes a language to describe reality, and not only: it amplifies *understanding*.

A process of this kind occurs very slowly, and through progressive steps, with an intertwining of continuities and ruptures between the different levels of knowledge. Traditional arithmetic teaching tends to favour a mental attitude aiming at immediately searching for the tools (operations) to identify the *answer* (the result). A little example shows how the attitude we are talking about is induced by the very formulation of standard tasks, like the following (pupils aged 6):

> On a tree branch there are 13 crows. 9 more arrive and 6 fly away.
> How many crows are left?

This trend reminds of the child's initial aims, when he uses language to satisfy his *primary* needs (hunger, sleep, pleasure, …). Growing up, he learns that verbal language fulfils richer and different functions, getting into the complexity of its structures, starting up his way toward knowing himself and the functioning of his own thought. Something similar should happen with arithmetic and algebra. The development of arithmetic thinking, characterized by operations on known numbers, may cause the rise of stereotypes in pupils that are difficult to eradicate. Due to these stereotypes, the student might get stuck in the obsessive search for the numeric result (*the primary need*) and not explore different and more fruitful mental routes that may lead to an embryonic algebraic thinking *(interpretation and description of reality* through mathematical language).

With this objective in mind, the text of the previous problem should be re-formulated so that even very young pupils might be enabled to reflect upon themselves making calculations (with the additional enactment of non-trivial argumentative skills):

> On a tree branch there are 13 crows. 9 more arrive and 6 fly away.
> Represent the situation in mathematical language, in order to find the number of the crows left.

The first version emphasizes the search for the *product* (16), the second one the search for the *process* (13+9–6), i.e. of the representation of the *relations* between the involved elements. This difference links back to one of the most important aspects of the epistemological gap between arithmetic and algebra: while arithmetic requires an immediate search for the solution, algebra postpones this and starts with a formal transposition of the problem situation from the domain of natural language to a specific representation system (think about a problem that can be solved by an equation).

The perspective of an approach to algebra as a language, in a continuous back and forth of thinking from arithmetic to algebra and vice versa, fosters a more effective teaching with pupils aged between seven and fourteen, characterized by *negotiation* and *explicit statement* of a didactical contract for the solution of problems, based on the principle: *"first represent, then solve"*. This seems extremely promising for dealing with one of the most important key elements in the conceptual field of algebra: the transposition in terms of representation from natural language, in which problems are either formulated or described, to the formal-algebraic one in which the relations and later the solution are translated.

Representation leads us to another key point in our theoretical framework: the syntax and semantics of mathematical language.

5.3 Respecting the rules between syntax and semantics

An aspect that strictly links to that of representation is the *respect of the rules* in the use of a language, even more necessary when one deals with a formalized language, given that the symbols used are extremely synthetic.

In everyday life the respect of linguistic rules is gradually learned through their use, by trial and error; this is favoured by the family environment and by the enlarged social one, as well as by school, through a reflection upon orthography, grammar, syntax, i.e. upon the structural aspects of a language.

In the learning of mathematics, the rules are generally 'given' to pupils, thus losing their social value as support to understand a language, and hence to share it, as a tool for communication. Similarly to what happens in linguistics, the syntax of the mathematical language concerns the structure of the sentence, the elements that compose it and the formal procedures that express the

relations between the involved quantities – either known or unknown- even in a period made of several sentences.
It is thus necessary to lead students to understand that they are appropriating a new language and that, as all other languages elaborated by the human kind, it is a system of arbitrary finite symbols, combined according to precise rules. But, whereas a pupil interiorizes since his birth that the set of rules related to spoken language, and respecting them, is functional to *communication*, it is rather hard for him to transfer this peculiarity to the mathematical language. To avoid this key element and highlight the value of written language for communication, the teacher proposes an exchange of messages in arithmetic-algebraic language with either real or virtual classes, engaged in the solution of the same problem situation. Brioshi, a virtual Japanese pupil, variably aged depending on the age of his interlocutors, knowing only his mother tongue (and therefore not able to communicate using languages that differ to his own), but competent in the use of mathematical language, is the *algebraic pen friend,* with whom they need to communicate (for a wider discussion, see Malara & Navarra 2001).
Pupils get to learn that, like any language, also the mathematical language has its own grammar and a syntax, i.e. a set of conventions that enable us to construct sentences correctly. It has a syntax, which provides the conditions – i.e. the rules – to decide whether a sequence of linguistic elements is 'well-formed' (for example, sentences like '9++6=15', or the classic chain of operations added one after the other, like '5+3=8:2=4+16=20') are syntactically wrong. It has a semantics, which enables one to interpret symbols – within syntactically correct sequences- and subsequently decide whether the expressions are true or false (for instance, the sentence '1+1=10' is either true or false depending on the representation base, which can be either 2 or 10).
In the perspective we are considering, *translating* from natural language (or graphical, or iconic) to mathematical one, and vice versa, is one of the most fertile territories where reflections upon mathematical language can be developed. Translating, in this case means interpreting and representing a problem situation through a formalized language or, on the contrary, recognize in a symbolic expression, the situation it describes.

In the learning of mathematics, where the exchange between verbal language and mathematical language is continuous, it is necessary to activate in pupils, on the one hand, a control over expressive registers and, on the other hand, the meta cognitive skill to understand how syntactic transformations of formal expressions condense thinking processes that can be hardly realized through natural language.
In the background, there is a delicate point, that we will deal with in the following, linked to the different, and often not made explicit, meanings of the sign '='.

5.4 Canonical/non canonical form of a number and the sign '='

Facing the question: 'Is $[3\times(11+7):9]^2$ a number?' usually both students and trainee teachers answer in the same way: "No, they are operations", "It is an expression", "They are calculations". At times, someone dares to say "Can we say it is the representation of a number?". To promote a reflection on the answer, as well as on the fact that it is not a case if the first numbers of our life, those that leave a sort of conceptual imprinting, have been called *natural*, we make use of the following strategy: we ask a pupil his name, his mother's, father's names, if he has siblings, if he has pets, his address, the number of his mobile phone, if applicable, and so on.
After a while, the blackboard is filled with a list similar to the following:

Marika
Laura's daughter
Matteo's daughter
Christian's sister
Renato's niece
Owner of the dog Floppy
Living in 24, Mr. What's-his-name street
…

The aim is to find out with the whole class that these are ways to *describe* the pupil: Marika is her first name and all the other definitions (we might call them representations) enlarge our knowledge of Marika, adding information not included in her first name.

We then carry on by explaining that the situation is similar with numbers: each number can be represented in many different ways, through any odd equivalent expression. Among these representations, one (for instance 12) is its name, called *canonical form*, all the others (3×4, (2+2)×3, 36/3, 10+2, …) are its *non canonical forms*, and each of them will make sense in relation to the context and the underlying process.

This experience enables elder pupils to answer the initial question we left unanswered: $[3\times(11+7):9]^2$ *is one of the many non canonical forms of the number 36*.

Being able to recognize and interpret these forms, builds up the semantic basis for the understanding of algebraic expressions like -4p, ab, x^2y, k/3. The process through which these skills are constructed is very long and is to be developed along the whole course of the first school years.

The concept of canonical/non canonical form has crucial implications for the pupil (as well as for the teacher), to reflect upon the possible meanings attached to the equality sign.

In 6+11-2=15, for example, both teachers and pupils often 'see' spontaneously *operations* on the left hand side and a *result* on the right hand side of the equal sign. The main idea is: 'I sum up 6 and 11, then take away 2 and get 15'. It is clear from this that there is another imprinting, pointed out by Kieran (1981, 1989, 1992), similar to that related to natural numbers: in the usual teaching of arithmetic in the first seven years of school, the equal sign has for the pupil the meaning of *directional operator*, and it has a space-time characterization. It prepares the conclusion of a story (calculations) which is to be read from the left to the right, up to its conclusion (the result).

In the move to algebra, the equal sign acquires another meaning; in an expression like 2a-6=2(a–3) it gets a *relational* meaning: it points to the equivalence between two representations of the same quantity (it relates the two expressions to each other, and embeds an idea of symmetry between them). The student must learn to move around in a conceptual universe, where it is necessary to go beyond the familiar space-time characterization. He is not generally alerted about the fact that he has to do with wider meanings; if his rooted conception is that 'the number after the equal is the result', an

expression like '$-7=n$' will probably mean very little to him, even though he might be able to solve the linear equation that leads to it.
An example of this: the request 'Write down 14 plus 23' very often in primary school gets the answer '14+23='. The equal sign is viewed as an *indicator of a conclusion* and expresses the implicit belief that this conclusion will sooner or later be required by the teacher. '14+23' is viewed as an event waiting for its realization. An operative attitude prevails, fruit of a kind of teaching that up to that moment has been focused on calculations. As stressed by Kieran, the fact that the sign '=' is missing, is viewed as if the operation would not be closed, as if the expression 14+23 (without equal) would be 'defective'. The student is a victim of a lack (or rather a poverty) of control over meanings.
Often, facing issues like the representations of a number and the meaning of the equal sign, Italian teachers are defenseless on the epistemological level. Once again, the important role of teacher training comes out strongly.
So far, we have been analyzing the theoretical setting of our conception of early algebra. We will now move on to the side of practice, with reference to both (i) *the classroom based action*, and (ii) *the teacher's reflection upon the latter*.

6. From theory to practice

Instruments, methods and activities outlined and tuned in the ArAl project, work as a support for teachers to propose early algebra activities in the classroom, using a socio-constructive methodology, and, at the same time, as a training to become *metacognitive teachers* through a reflection upon their own action in the classroom. Follow-ups of this are twofold:

- on the pupils' side: the aim is to analyze the conditions under which pupils, since grades 4-6 manage, at a first level, to generalize, formulate properties and produce formal representations and, at a second level, to appropriate the meaning of algebraic expressions and become aware of their expressive strength;

- on the teacher's side: the aims are on two levels as well. One aim is to refine their ability to guide the class in the approach to early algebra following these modalities; a second aim is to foster their professional development through stimuli deriving from participation in at least two-year collaboration projects, characterized by the immersion in a community of enquiry on one's own practice, in a continuous interplay of *reflection*, *exchange*, *sharing*.

We will get back to all these aspects in the next paragraphs; let us now deal with the basic instrument of the educational project for both class activity and teacher training: the ArAl Units[3].

6.1. The Units

The Units, monographic booklets about experiences in early algebra, are models of processes of teaching arithmetic in an algebraic perspective, to be carried out in the long term. Before publication, they are refined through a three-year-long process, characterized by a progressive refinement of teaching experiments in different classes, analysis of the related audio recordings, reflection exchanges among university researchers, in the framework of a methodology that will be illustrated in one of the next paragraphs.

The units (see figure 1) are structured so that they describe – in the left-hand side of each page – a synthetic sequence of teaching pathways and, at the same time, they clearly show - in the right-hand side column –those aspects, drawn from an analytical reading of the class transcripts, which can help teachers to apply it (methodological choices, enacted class processes, key elements of the processes, extensions, potential behaviours of pupils, difficulties they might meet and so on). We will get back to class transcripts in one of the next paragraphs.

[3] Eleven Units have been published in Italy up to now (2009) by Pitagora Editrice, Bologna (http://www.pitagoragroup.it/pited/progettoARAL.html). For eight of them there is an English version (http://www.aralweb.unimore.it/on-line/Home/ArAlProject/Download.html).

The overall objective of the units is to provide teachers not only with teaching sequences, but also with spaces for reflection on their own knowledge and on their way of working in the classroom.
The situations concern explorations on: grids of numbers, towers of number blocks, multi-coloured necklaces with various modules of bids, situations of balance on a scale, games of translation between natural and formal language, etc. Due to these exploratory activities, pupils approach the use of letters, enact the construction and elaboration of the first algebraic expressions, construct and solve equations reflecting on the underlying processes, interpret the meaning of formal expressions with relation to specific issues. They thus acquire an algebraic way of looking at arithmetic.
The situations are dealt with in stimulating, but not easily manageable, teaching and learning environments, and imply for teachers a number of delicate aspects involving several skills. They come to take into account their own knowledge and convictions, and at the same time, a surrounding made of non secondary methodological and organizational aspects, which operatively support a culture of change.

6.2. The methodology

Noticing and critically studying and reflecting upon classroom-based processes allow the teacher to become aware of the processes enacted in the teaching activity, as well as of the variables that determine them.
With these aims, we outlined the *Multi-Commented Transcripts Methodology*, MCTM, based on the critical analysis of transcripts of whole-class discussions, carried out by the teachers, through the intervention of a variable number of actors: the class teacher, his/her E-tutor, other teachers engaged in the same teaching experiments, teachers-researchers and university researchers. This is the core of our way of working with and for teachers, which envisages also periodical meetings for a critical exchange among teachers and among teachers, mentors and researchers on the work progressively done.

We are going to illustrate the MCTM, enriching it with an example that will enable us to point out the correlation between the difficulties met by pupils and the teacher's attitudes.

6.3 The multi-commented transcripts methodology

The MCTM's aim is to lead teachers to acquire an increasing capacity of interpreting the complexity of class processes through the analysis of the *micro-situations* that constitute them, to reflect upon the effectiveness of one's own role and become aware of the effects of one's own *micro-decisions*.

It is structured in a sequence of phases. Teachers experimenters make audio-recordings of lessons on topics they previously chose in agreement with researchers and, after transcribing them in digital text version (the 'transcripts') and filling them with comments and reflections, they send them to their E-tutors, for further comments. Then, the latter send the transcripts to the authors, to other teachers engaged in similar activities and sometimes to other researchers. Often the authors intervene back in the cycle, making comments about the comments or rather inserting new ones.

Multi-commented transcripts are important instruments from four points of view:

- Diagnostic: transcripts provide the E-tutor, and therefore the whole team, with an overview of the teacher's teaching action and enable a check on the coherence between teaching practice and reference to the theory at stake (both mathematics and mathematics education).
- Formative: through comments, they enable the teacher to develop competences and sensitiveness and hence to improve the overall quality of his teaching action.
- Evaluative: by making clear both coherence and inconsistencies of the teaching action, the transcripts provide both the teacher and the researchers with elements that can empower the effectiveness of the interventions in the respective areas, and make it possible to detect teachers' attitudes and cultural backgrounds.

- Social: the transcripts favour a sharing of knowledge, since they are sent out to the other components of the group and a periodic reflection upon the most significant excerpts is undertaken. Moreover, each teacher, comparing his own progress on the realization of a certain part of the teaching sequence to that of other colleagues, can identify important distinctive elements and reflect upon both effectiveness and limits of his own work.

Taking part in the MCTM the teacher questions his action at different levels:

- transcription promotes a posterior reflection upon the activity and how it has been carried out and guided;
- writing down the comments fosters a critical reconstruction of the activity through an interpretive effort which has a high formative value;
- their analysis by E-tutors – focusing on both mathematical and methodological aspects – leads to a re-elaboration of the activity with a significant impact on both teaching practice and teacher training.

The methodology enables the teacher to critically connect three key points: his own conception of the mathematical knowledge at stake (and of mathematics itself); the conflict induced by the meeting-clash with the teaching modalities enacted by colleagues and the results they achieved; the mediation between these two key points produced by the collective exchange and the dialogic relationship with researchers.

The comments (almost line by line) make clear the overall analysis which deals with aspects related to content (the approach to the exploration of the problem, the mathematical aspects developed, the objectives achieved or missed out, …), aspects related to communication and language used (formulation of the posed questions, interlocutory expressions used, operative directions suggested, …), issues related to the control over pupils' participation (number and type of interventions), as well as to the didactical contract established (attitudes induced in pupils and socio-mathematical norms in the classroom). The commented transcripts come to be a real 'radiography' of the teacher, facing which the latter might go through healthy critical moments, almost always

followed by positive reactions of self challenge, which lead him to act towards a requalification as a professional.
The multi-commented transcripts offer wide materials for the construction of prototypes of activities of critical reflection, used in both laboratories and training of didactics of the disciplines, in-service and pre-service courses, in the construction of learning objects, in the specific training of mentors and supervisors.

6.4. From the comments to a classification of attitudes

As we repeatedly underlined, the aim of the project is to train metacognitive teachers. In this view, the high number of comments in the transcripts (up to 2009 nearly 4000) offers a very meaningful overview of *the teachers' attitudes* towards their own activity in the classroom and their own capacity of reflecting upon it afterwards. The analysis of comments, in turn, brings about a powerful feedback on training interventions.

Let us assume two categories, both related to the teacher and his 'being metacognitive':

(a) in his action in the classroom;
(b) in the posterior reflection upon his action in the classroom.

The two categories enable us to identify four different types of attitudes, summarized in the table (Fig. 2):

Fig. 2: Types of attitudes at metacognitve level

		Teachers	
		M_C metacognitive in comments	$\neg M_C$ non metacognitive in comments
Teachers	M_A Metacognitive in action in the classroom	They stimulate meta cognitive attitudes; *they comment upon transcripts in depth.*	They stimulate meta cognitive attitudes; *they insert few meaningful comments in diaries.*
Teachers	$\neg M_A$ Non meta cognitive in action in the classroom	They do not stimulate meta cognitive attitudes; *transcripts are filled in with posterior reflections upon missed out chances.*	They do not stimulate meta cognitive attitudes; *they insert in transcripts few meaningful comments.*

The four deriving profiles may be defined as follows:

$M_A M_C$ The teacher effectively drives the class discussion towards the mathematical objective, encouraging pupils to explore the problem situation; he tries to adjust to
When the session is transcribed, he keeps detached from the events and this leads him to write either theoretical or practical reflections both with relation to mistakes – stiffness, wrong interpretations – and to fruitful interventions.

$M_A \neg M_C$ The teacher manages the activity with a global self-confidence, thus favouring a reflection upon the processes, the coherence in pursuing the objective, the exchange among peers. In the transcription phase, he does not detect stimulating moments for reflection and views his task as essentially completed in the classroom activity. In other words, the phase of in-depth analysis and generalization of behaviours with relation to theoretical issues seems to be weak.

$\neg M_A M_C$ The teacher – due to scarce expertise, low control over mathematical contents, difficulties in managing the discussion– is not able to guide the activity in a productive way, stimulating the attention of the class towards a meaningful reflection. At the moment of transcription, he realizes how weak his guidance was (a feeling often emerged at the very moment of its appearance) and the comments point out this awareness, often joint to a request of help addressing the e-tutor.

$\neg M_A \neg M_C$ The teacher keeps the class working on a little stimulating activity, constantly playing a central role and driving the pupils towards a fundamentally a-critical acquisition of ‘compulsorily reachable’ contents. In the phase of transcription, the comments refer to marginal aspects, concise remarks on pupils' personality, superficial clarifications.

In general terms, the type $M_A M_C$ represents a desirable model and possible target of any formative process. In fact the teacher is often not used to an exchange in which he plays the role of the student, with hardly predictable consequences.

This is confirmed by a teacher who, after a two-year long collaboration writes down in his reflections:

> "At the beginning the Glossary and the ArAl Units were my points of reference. I started to produce my first transcripts rather timidly. Through these, and helped by the comments of my E-tutor and another researcher, I was able to analyze my behaviours and those of my pupils in teaching and learning situations, focusing on both positive and negative aspects, and to reflect upon some crucial points, mainly related to the management of the mathematical aspects and to communication. And that was the beginning of an itinerary."

Often, an $M_A \neg M_C$ – type attitude is not a sort of 'limited disposition' to reflection, but rather fruit of the lack of meaningful stimuli towards the direction Mason talks about, of a constant search within a process and towards oneself, which may leave deep and long-lasting traces at the professional level, contributing to the construction of a way of being that will become *founding trace of change*.

The definition of attitudes $\neg M_A$ is more complex. They are also linked to the age, and hence expertise, of the teacher, but strongly reflect his personality (low self-confidence, fear of losing control over the class, tendency to keep to a reassuring professional stability, tiredness and so on) and his personal history (education, limited attention to refresher courses, skepticism towards theories viewed – sometimes correctly- as too abstract etc.). In these cases the formative intervention steered towards early algebra becomes powerful because it does not aim to *add* some knowledge but rather attempts to induce a *reflection* which might prepare the ground for a *restructuring of knowledge*, and most of all, might promote the

construction of a mental attitude open to new perspectives concerning both theory and practice.
The four profiles are not *pictures of teachers*, in the sense that they are not definitions to be taken as *absolute*. They rather illustrate *temporary stages* at which the teachers who adopt the multi-commented transcripts methodology come to be and which can be modified over time. The improvement in the capacity of expressing metacognitive attitudes can thus be seen as the outcome of a formative process to which researchers contributed – we hope significantly– but it will inevitably be up to the teacher to make it grow autonomously during his professional activity, if he decides to.

The following example is taken from a transcript of a teacher in the phase $\neg M_A M_C$. The example highlights the negative incidence of the teacher's language on the discussion: this language is not appropriate and pays little attention to the aims of the work which is being done. The comments are in this case made by the teacher himself and by three e-tutors, but after some time, the teacher presented excerpts from his transcripts at a conference, enriching them with personal remarks- often in the form of meta-comments – which highlight how the methodology had positively influenced his attitude, leading it towards a model $M_A M_C$.

The example was chosen because it shows the value of the methodology we adopted, to promote teachers' awareness of the limits of their own action. We realize that this example alone cannot fully transmit the richness and variety of the emergent issues, as well as what the teachers come to achieve; for this reason we send the reader back to Malara (2008).

Example

The example refers to a moment of a teaching sequence about the study of sequences that can be modelled algebraically. The main objective of the sequence is to make pupils achieve a functional view of sequences and lead them to construct algebraic representations for the latter, modelling the relation between the ranking (or place) number and the correspondent term of the sequence. The teacher had proposed the exploration of the sequence after living the first three terms: 4; 11; 18; … . The class had grasped that the sequence was generated, starting from 4, by the operator "+7". Then the teacher had raised the question of searching for a formula that could represent the correspondence <place-term>. In the classroom, attention was then focused on the problem of generalization and the whole class had worked on the meaning of the term ‘n-th’. To make the pupils’ exploratory activity easier the teacher had summarized in a table all the they knew at that moment. In the study of the case at place 30th there was a mistake: the number before 30 is swapped with the number after it, during the generalization phase.

Place	Number	Operations	Rule 1	Rule 2
1°	4			
2°	11	4+7	4+(7x1)	7x3–10
3°	18	4+7+7	4+(7x2)	7x4–10
4°	25	4+7+7+7	4+(7x3)	7x5–10
30°			4+(7x31)	
n				

Teacher: *I want to know: if I am at place n, that we said – do you remember? – it was a place at a certain point, without knowing what point it was. Eh, I want to know what is the rule that allows me to find this number at place n* **(1)** [I point to the n-th term on the blackboard] *are you with me*?

Teacher: *Good, so let's find the rule* **(2)**. *Benedetta*?

Benedetta: *Eh, because I believed that n-th was the last, so I wrote " there isn't because the sequence is infinite* **(3)**".

Teacher: *All right, this is a true remark and perhaps it will be useful later, we will keep it. So, how can we find the formula we need? Don't look at me, look at your sheet and the blackboard! How can you find it?* **(4)** *Andrea?*

Andrea: *Now, if we know… last time we said that n-th stood for any place* **(5)**

Teacher: *Question: number n means a number at any place* **(6)** *without saying what number it is, this is the hard part! What formula do I write for the number at the n-th place*? **(7)**

Sergio: *Er… I think you can't find it because n-th is a number you don't know*

Andrea: *As you said, n-th stands for a number at any place, therefore I say with Sergio, if the place is indefinite, we will never get to know what number it is!* **(8)**

Teacher: *Exactly, I agree with you too! If I don't tell you, at the 3rd, at the 4th, at the 100th, at the 7003rd place, you don't know. But if I tell you that this number … about this number, instead of telling you the place number I tell you it is at place n, can I calculate … can I write a formula to write this number?* **(9)**

Comments

(1) **T**[4] I now realize I have used the wrong terms, thus inducing students to give the answers that will follow, and that I desperately fought against. By saying "the rule to find this number at place n" the students understood that I wanted to know that value of a_n. Perhaps I should have said "the rule to find a number of the sequence, given its position".

(2) **M2** I suggest that the class be led to discover and highlight with arrows relations, repetitions of numbers, 'local' regularities. Many of these might be not productive, but they help pupils get used to global explorations. For example, the same sequence, proposed in another class, led some pupils to identify a relation between the n umbers of the first two columns and to represent it with $11 = 2 \times 7 - 3$, $18 = 3 \times 7 - 3$, $25 = 4 \times 7 - 3$, and so on. The arrows might link the various four's with the first term of the sequence, numbers 1, 2, 3 of the fourth column with the place numbers of the first one, shifted one line down, etc. These two last arrows might show that 31 is wrong and that it should be substituted for 29.

(3) **M3** Benedetta contradicts herself, if the sequence is infinite, also the places are infinite and n cannot be the last one. Perhaps she means n as 'very large' number. With this contradiction she expresses her belief that a number at a non-defined place cannot be represented. Anyway, she does not know the meaning of n as indicator of a number we do not want to state explicitly.

(4) **M1** T is worried by the idea that she should get to the formula written in algebraic language. I keep believing that, in this phase of the work, the objective is to lead pupils to grasp the relation between place and correspondent number, and to express that relation clearly.

M2 Why not encouraging expression in natural language, describing the forms of columns 3 and 4: "I get the number by adding to the initial number as many 7 as …" or in any other way. The paraphrases proposed by pupils can then be compared and the most suitable to be translated in algebraic language for Brioshi can be chosen..

(5) **M1** Well done Andrea, that "any place" is like gold!

(6) **M3** More than 'any', term which is linked to the idea of variable, it would have been appropriate to underline that it is a number we do not want to specify, 'indeterminate' (term which, focusing on the element, fixes it somehow)

(7) **M1** I wondered many times: why not putting, in the column with the mathematical sentence, the (either mental or not) operation made to identify the factor that multiplies 7, starting from the given number? Pupils would have grasped the regularity, the reiteration of a procedure, getting closer to the construction of the formula smoothly.

---

[4] Here onwards, we will use the following codes:
T = teacher; M1 = mentor 1; M2 = mentor 2; M3 = mentor 3.

M3. The formula is gradually determined by identifying invariant parts (4+ 7x…) and variant parts (number of place – 1) in the studied cases.

(8) **M2** the approach to the letter is rather complex, requires lengthy times, different strategies, comparisons, explorations, entails continuous and unpredictable evaporations. The joint presence of intuitions of different meanings in the interventions of Sergio and Andrea is absolutely inevitable, almost physiological. Probably the need (real or presumed) to conclude and get to the rule, imposes to the teacher rhythms that can hardly coexist with that complexity. We are fully immersed in algebraic babbling, and the learning of a new language, of its meanings and rules, must respect the needs of a required settling.

(9) **T** Now I see why they could not answer! We don't understand each other! As I said at the beginning, the verb "to find" puts them on the wrong track! Perhaps I should have said "find a representation of the number at place n which makes us understand that this number is in the sequence". Too complicated! I don't know…

M1 I agree on the damages caused by the term "to find".

M2 I also agree on representing, even more if this term (Glossary) becomes one of the keywords of the class' cultural background, and hence acquires a shared and negotiated meaning (Glossary again).

M3 Finally, well done T! Representing, yes, representing is the keyword.

The analysis of the comments clearly shows the epistemology of the researcher who produced them, due to the prevalence of some types of comments. Both agreements and disagreements in these comments turn out to be fruitful for the teacher, the former by reinforcing the comment, the latter as enriching complements.

7. Concluding remarks

Let us examine what we discussed so far. We said that, for most students, the big obstacle in the study of algebra is represented by the difficulty in having control over the meanings of formal expressions. They are led to the manipulation of the later through the application of rules that are semantically opaque. This is the main reason why in K-8 stages' curricula increasing space is given to early algebra, in association with a socio-constructive teaching practice. The aim is to propose a kind of teaching that may *revisit arithmetic in a pre-algebraic perspective* introducing in primary school activities that foster the development of pre-concepts useful for the learning of algebra.

Within this frame work our investigations (ArAl Project) are carried out, based on the conviction that *the main cognitive obstacles in the learning of algebra arise in arithmetical contexts and might condition the development of mathematical thinking* in students, due to a weak conceptual control over the *meanings* of algebraic objects and processes. Some of its most important principles are: anticipation of pre-algebraic activities of a generational type, social construction of knowledge, central role of natural language as didactical mediator, identification and explicit expression of algebraic thinking, often 'hidden' in arithmetical concepts and representations.
As a consequence, the issue of teacher training come to be crucial for our aims. In this respect, we underlined the value of *critical reflections upon classroom-based processes,* also through participation in 'communities of enquiry' made of teachers and researchers, like those involved in the ArAl project[5]. Starting up a continuous reflection upon oneself as a professional in education, implies that one understands the *directions* he should go to support transformation through an *inter-exchange between theory and practice.*
We pointed out how, from our point of view, early algebra defines its area of interest starting from both disciplines (either arithmetic or algebra) and gets a different, and mainly original, identity. We have defined early algebra as a *meta discipline,* dealing not only with entities, processes and properties of the two subjects, but rather with the genesis of a unifying language and, therefore, of a metalanguage. A Glossary supports the construction of this metadisciplinary knowledge. We illustrated some key constructs: *algebraic babbling*, the pair *solving* and *representing*, *canonical* and *non canonical* form of a natural number, the *equality* sign, the respect of the rules in the approach to the *algebraic code*, syntax and semantics, Brioshi. The Units are the basic instrument of the whole educational project.

Finally, we illustrated the Methodology of Multi-commented

[5] Since 2000 nearly 1000 teachers and more than 10.000 pupils from 12 Italian regions participated in the project.

Transcripts and its central role in the teacher's formative process, to empower his capacity of reflection, as well as to construct a constant attitude of noticing his own behavior in the classroom-based action and to have control over the impact that his way of acting may have on pupils' attitudes and conquests.

With all this, perhaps it is still very difficult for a teacher – even a good teacher- to make students become passionate protagonists of their own learning. It would be a lot if he/she could solicit curiosity, reduce difficulties, smooth fears, promote reflections. In other words, if he/she could open up more and more daring views for pupils, so that they become able to investigate on their own individual inclinations and understand how much intellectual and emotional energies they really want to employ in the intellectual challenges they face.

In the field of mathematics, taking students towards this target means for teachers, researchers and trainers, asking oneself some key questions. Among these, the main one may be introduced through a metaphor taken from the theatre's world: *when does the curtain start being raised on algebra?*

Some examples.

- Anna (final year at kindergarten) recognizes that two trains that continue who-knows-where beyond the room's door - one made of wagons with two yellow and one red blocks and the other made of wagons with two nuts and a seed of sunflower – "are almost equal". Is Anna doing some algebra?
  In fact she plays with *structural analogy.*
- Federica (grade 2, primary school) finds on her book the expression '$3\times\square=27$' and writes down '$3\times9=27$'. The teacher tells her she was good because she knows the times tables. Is Federica doing some algebra?
  In fact *she solved a linear equation with one unknown*.
- Piero (grade 3, primary school) notices that "It is correct to say that 5 plus 6 is 11, but you cannot say that 11 'is' 5 plus 6, and then it is better saying that 5 plus 6 'equals' 11, because in this case the contrary is true as well". Is Piero doing some algebra?
  In fact he is arguing on the *relational meaning of equal*.

Introducing pupils aged between 5 and 14 to mathematics in the perspective of early algebra essentially means to guide them– through suitable problem situations dealt with in a socio-constructive context – towards a new language, provided with its own syntax and semantics, in which respecting the rules is constraining, so that they might be able to: *translate, argue, interpret, predict, communicate*. [6] Making calculations is still there, but subordinated to 'higher' purposes, it helps prepare reasoning, argumentations, refutations, corrections. All this will lead pupils to understand, as the complexity of the algebra they will deal with increases, that manipulation of symbols (polynomials, equations, functions etc.) is not self-referential, but helps pupils mathematize, explore, reason, deduce, in other words, produce thinking and achieve new knowledge.

As we said earlier, our main goal, of a 'meta' type, is to form metacognitive students. But to do this, it is necessary that teachers learn to be metacognitive teachers in turn. We examined instruments and methods we outlined to promote metacognition in teachers, in a strict intertwining of reflections upon knowledge at stake (theory) and action in the classroom (practice) and showed the value of an educational process which, in the long term, is able to give them a new professional identity, more consistent with the role they need to play, and not only with reference to early algebra. This is a condition for inducing in pupils, since the early school years, a view of algebra as a language and as an instrument for thinking, a constructive and

---

[6] For example:

- *Translating:* 'Represent the following sentence using the mathematical language: Add a number to the triple of 7 and you get 26';
- *Interpreting:* 'Is 578×3+2 a multiple of 3? Justify your answer'
- *Predicting:* 'Three friends - a hare which makes 5 units-long jumps, a frog that makes 3 units-long jumps and a cricket, that makes 2 units-long ones – start from the same stone and head to the same place. Do they, early or later, get to the same point of the trail? Justify your answer'
- *Communicating:* Exchange of messages between Andrea's class and Brioshi:

  Brioshi 16=a+5-2
  Andrea's c. 16-5-2=a; 9=a
  Brioshi 9≠a!!!
  Andrea's c. 16-5+2=a; 13=a

reflexive attitude and, more in general, a conception of mathematics as carrier of meanings.

References

Ainley, J. (coord.). (2001). Research Forum on Early Algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 128-159). Utrecht (The Netherlands).

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9, 33–52.

Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.

Arzarello, F. (1991). Procedural and relational aspects of algebraic thinking. In F. Furinghetti (Ed.), *Proceedings of the 15th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol.1, pp. 80-87). Assisi (Italy).

Arzarello F., Bazzini L. and Chiappini, G. (1993). Cognitive Processes in Algebraic Thinking: Towards a Theoretical Framework? In I. Hirabayashi, N. Nohda, K. Shigematsu, and F. L. Lin (Eds.), *Proceedings of 17th International Conference on the Psychology of Mathematics Education*, (vol.1, pp. 138-145). Tokio (Japan).

Bell, A. (1976). A study of pupils' proof explanations, *Educational Studies in Mathematics*, 7, 23-40.

Bell, A., Swan, M., Onslow, B., Pratt, K., and Purdy, D. (1985). *Diagnostic Teaching for Long Term Learning. Report of ESRC Project HR8491/1*. Shell Centre for Mathematical Education, University of Nottingham.

Blanton, M. and Kaput, J. (2001). Algebrafying the elementary mathematics experience: transforming task structures: transforming practice on a district-wide scale. In E. Chick, et Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of Algebra'* (vol. 1, 87-95). Melbourne (Australia).

Blanton, M. and Kaput, J. (2002). Design principles for tasks that support algebraic thinking in elementary school classroom. In A.D. Cockburn and E. Nardi (Eds.), *Proceedings of 26th International Conference on the Psychology of Mathematics Education* (vol. 2, pp. 105-112). Norwich (UK).

Blanton, M. L., Stylianou, D. A., and David, M. M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. In: N. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of 27th International Conference on the Psychology of Mathematics Education* (Vol. 2, pp. 113-120). Honolulu (Hawaii).

Borasi, R., Fonzi, J., Smith, C. and Rose, B. (1999). Beginning the Process of Rethinking Mathematics Instruction: a Professional Development Program, *Journal of Mathematics Teacher Education*, 2(1), 49-78.

Booth, L.R. (1984). *Algebra: Children's strategies and errors. A report of the strategies and errors in Secondary Mathematics Project*. Windsor, Berks.: NFER-Nelson

Cai, J., Lew, H. C., Morris, A. , Moyer, J. C., Ng, S. F. and Schmittau, J. (2005). The Development of Students' Algebraic Thinking in Earlier Grades: A Cross-Cultural Comparative Perspective, *ZDM*, 37(1), 5-15.

Carpenter, T. and Franke, M. L. (2001). Developing algebraic reasoning in the elementary school: generalization and proof. In E. Chick et Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of algebra'* (vol. 1, pp. 155-162). Melbourne (Australia).

Carpenter, T.P., Franke, M.L. and Levi, L. (2003). *Thinking Mathematically. Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Carraher, D., Brizuela, B. and Schliemann, A. (2000). Bringing out the algebraic character of Arithmetic: instantiating variables in addition and subtraction. In T. Nakahara e M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 145-152). Hiroshima (Japan).

Chick, E., Stacey, K, Vincent, Jl. and Vincent, Jn. (Eds). (2001). *Proceedings of the 12th ICMI Study 'The future of the teaching* *and learning of Algebra'*, Melbourne, Australia: University of Melbourne.

Cusi, A. (2008). An approach to proof in elementary number theory focused on representation and interpretation aspects: the teacher's role. In B. Czarnocha (Ed.), *Handbook of Mathematics Teaching Research* (pp. 107-122). Rzeszów University Press.

Cusi, A. and Malara, N. A. (2008). Approaching early algebra: Teachers' educational processes and classroom experiences. *Quadrante*, 16(1), 57-80.

Da Roca Falcão, J.T., (1995), A case study of Algebraic Scaffolding: from Balance to algebraic notation. In L. Meira and D. Carraher (Eds.), *Proceedings of the 19th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, 66-73). Recife (Brazil).

Dougherty, B (2001). Access to algebra: a process approach. In E. Chick et Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of algebra'* (vol. 1, pp. 207-212). Melbourne (Australia).

DFE (Department for Education) (1995). Mathematics in the National Curriculum, HMSO, London.

Ferreri, S. (2006). Parole tra quantità e qualità. In I. Tempesta and M. Maggio (Eds.) *Linguaggio, mente, parole, Dall'infanzia all'adolescenza* (pp. 19-25). Collana GISCEL. Milano: Franco Angeli Editore.

Filloy E. (1990). PME algebra research. A working perspective. In G. Booker, P. Cobb and T. N. Mendicuti (Eds.), *Proceedings of the 14th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol.1, pp. 1-33). Oaxtepex (Mexico).

Filloy E. (1991) Cognitive tendencies and abstraction processes in algebra learning, In F. Furinghetti (Ed.), *Proceedings of the 15th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol.2, pp. 48-55). Assisi (Italy).

Gray, E. and Tall, D. (1993). Success and failure in Mathematics: The flexible meaning of symbols as process and concept. *Mathematics Teaching*, 142, 6-10.

Harper, E. (Ed.). (1987-88). *NMP Mathematics for Secondary School*, Essex, UK: Longman.
Kaput, J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical constructivism in Mathematics Education* (pp. 53-74). The Netherlands: Kluwer Academic Publishers.
Kaput, J. (1995). A research base supporting long term algebra reform? In D. T. Owens, M. K. Reed and G. M. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting of PME-NA* (Vol.1, pp. 71-94). Columbus (OH): ERIC Clearinghouse for Science, Mathematics and Enviromental Education.
Kaput J., Blanton M. (2001). Algebrafying the elementary mathematics experience: transforming task structures. In E. Chick et Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of algebra'* (vol. 1, 344-353). Melbourne (Australia).
Kaput, J. , Carraher, D. W. and Blanton, M. L. (Eds.) (2007). *Algebra in early grades*. Mahwah (NJ). Lawrence Erlbaum Associates.
Kieran, C. (1981). Concepts associated with the equality symbol, *Educational Studies in Mathematics*, 12, 317-326
Kieran, C. (1989). The Early Learning of Algebra: a Structurale Perspective. In S. Wagner and K. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 33-56). Reston (Virginia): LEA.
Kieran C. (1992), The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.
Kieran, C. (1996), The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde and A. Perez (Eds.), *8th International Congress on Mathematics Education: Selected Lectures* (pp. 271-290). Seville, Spain: S.A.E.M. Thales.
Kieran, C. (2004): Algebraic thinking in the early grades: What is it?, *The Mathematics Educator (Singapore)*, 8 (1), 139-151.
Kuchemann, D.E. (1981). Algebra. In K. Hart (Ed.), *Children Understanding Mathematics: 11-16* (pp. 102-119). London: Murray.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: a definition of pre-algebra. *Journal of Mathematical Behaviour*, 14, 113-120.

Lins, R. C. (1990). A framework for understanding what Algebraic Thinking is. In G. Booker, P. Cobb and T. N. Mendicuti (Eds.), *Proceedings of the 14th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol.2, pp. 93-100). Oaxtepex (Mexico).

Jaworski, B. (1998). Mathematics Teacher Research: Process, Practice and the Development of Teaching. *Journal of Mathematics Teacher Education*, 1, 3-31.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54, 249-282.

Jaworski, B. (2004). Grappling with Complexity: Co-learning in Inquiry Communities in Mathematics Teaching Development. In M.J. Hoines and A.B. Fuglestad (Ed.), *Proceedings of the 28th Annual Conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 17-36). Bergen (Norway).

Lee, L. and Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics* 20 (1), 41-54.

Malara, N.A. (2003). Dialectics between theory and practice: theoretical issues and aspects of practice from an early algebra project. In N.A. Pateman, B. J. Dougherty, and J. Zilliox (Eds.), *Proceedings of the Joint Meeting of PME and PME-NA*, (Vol. 1, pp. 33-48). Honululu: Center for Research and Development Group, University of Hawai.

Malara, N.A. (2005). Leading In-Service Teachers to Approach Early Algebra. In L. Santos et Al. (Eds.), *Mathematics Education: Paths and Crossroads*, (pp. 285-304). Lisbona: Etigrafe.

Malara, N.A. (2008). Methods and tools to promote a socio-constructive approach to mathematics teaching in teachers. In B. Czarnocha (Ed.), *Handbook of Mathematics Teaching Research* (pp. 89-102). Rzeszów University Press.

Malara, N.A and Iaderosa, R (1999). Theory and practice: a case of fruitful relationship for the renewal of the teaching and learning of algebra. In F. Jaquet (Ed.), *Proceedings of CIEAEM 50* (pp. 38-54). Neuchatel: Neuchatel University press

Malara, N.A. and Navarra, G. (2001). “Brioshi” and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language. In E. Chick et Al. (Eds.), *Proceedings of the 12th ICMI Study ‘The future of the teaching and learning of algebra’* (vol. 2, pp. 412-419). Melbourne (Australia).

Malara, N.A. and Navarra G. (2003). *ArAl project: Arithmetic Pathways Towards Favouring Pre-algebraic Thinking*. Bologna: Pitagora

Malara, N.A. and Navarra, G. (2008). An Early Algebra Glossary and its role in teacher education. In B. Czarnocha (Ed.), *Handbook of Mathematics Teaching Research* (pp. 193-207). Rzeszów University Press.

Malara, N.A. and Navarra, G. (2009). The analysis of classroom-based processes as a key task in teacher training for the approach to early algebra. In B. Clarke, B. Grevholm, and R. Millman (Eds.), *Tasks in Primary Mathematics Teacher Education* (pp. 235-262). Berlin: Springer.

Malara, N.A. and Zan, R. (2002). The Problematic Relationship between Theory and Practice. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 553-580). Mahwah (NJ): LEA.

Malara, N.A and Zan, R. (2008). The Complex Interplay between Theory and Practice: Reflections and Examples. In L. English (Ed.), *Handbook of International Research in Mathematics Education- II Edition* (pp. 539-564). New York: Routledge.

Mason, J. (1998). Enabling Teachers to Be Real Teachers: Necessary Levels of Awareness and Structure of Attention. *Journal of Mathematics Teacher Education*, 1, 243-267.

Mason, J. (2002). *Researching Your Own Practice: the Discipline of Noticing*, London: The Falmer Press.

Meira, L. (1990). Developing knowledge of functions through manipulation of a physical device. In G. Booker, P. Cobb and T. N. Mendicuti (Eds.), *Proceedings of the 14th Annual*

Conference of the International Group for the Psychology of Mathematics Education (vol.2, pp. 101-108). Oaxtepex (Mexico).

Menzel, B. (2001) Language conceptions of algebra are idiosincratic, in Chick, H. & Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of algebra'* (vol. 2, pp. 446-453). Melbourne (Australia).

NTCM (National Council of Teacher of Mathematics) (1998). *The nature and role of algebra in tthe K-14 curriculum*. Washington:DC: National Academy Press.

NCTM (National Council of Tecaher of Mathematics) (2000). *Priciples and Standards for School Mathematics*. Reston (VA): NTCM. http://standars.ntcm.org.

Ponte, J. P. (2004). Investigar a nossa própria prática: una stratégia de formaçáo e de construçáo de conhecimento professional. In E. Castro and E. De la Torre (Eds.), *Investigación en educación matemática* (pp. 61-84). University of Coruña (Spain).

Potari, D., and Jaworski, B. (2002). Tackling Complexity in Mathematics Teaching Development: Using the teaching triad as a tool for reflection analysis. *Journal of Mathematics Teacher Education*, 5 (4), 351-380.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics*, 42(3), 237-268.

Sherin, M.G. (2002). A balancing act: developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, 5, 205–233.

Shoenfeld, A. (1998). Toward a theory of teaching in context. *Issues in Education*, 4(1), 1-94.

Schwarz, B., Dreyfus, T., Hadas, N. and Hershkowitz, R. (2004). Teacher guidance of knowledge construction. In M.J. Høines and A.B. Fuglestad, *Proceedings of the 28th Annual Conference of the International Group for Psychology of Mathematics Education* (vol.4, pp. 169-176). Bergen (Norway).

Wood, T. (1999). Approaching Teacher Development: Practice into Theory. In B. Jaworski et Al. (Eds.), *Mathematics Teacher Education: Critical International Perspectives* (pp. 163-179). London: The Falmer Press.

Yackel, E. and Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.