EARLY ALGEBRA Y FORMACIÓN DEL PROFESORADO: EL CASO DEL PROYECTO ARAL

Early Algebra and Teachers Education: the case of the ArAl Project

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Resumen

Después de algunas indicaciones sobre los estudios que han determinado la génesis de la early algebra, presentamos el Proyecto ArAl – dirigido a la renovación de la enseñanza de la aritmética en perspectiva algebraica a través de un acercamiento lingüístico, socio-constructivo y metacognitivo. Discutimos acerca de aspectos teóricos del proyecto y de nuestras modalidades de trabajo con el profesorado para llevarles a adquirir formas de actuar en el aula en sintonía con los objetivos del proyecto. Presentamos un breve extracto de una discusión en clase que muestra la aparición de las primeras formas de pensamiento algebraico en los estudiantes y más en general, discutimos acerca de las dificultades identificadas en los docentes. Concluimos con observaciones sobre las condiciones que dan lugar a cambios en los docentes y la dificultad de averiguar en el tiempo su incidencia en el desarrollo de una nueva visión del algebra en los estudiantes.

***Palabras clave**: Early Algebra, Pensamiento Algebraico, Formación Profesorado, Relación Teoría-Práctica, Práctica reflexiva*

Abstract

After some indications on the studies which generate early algebra and on the socio-cultural aspects implied in its teaching, we present the ArAl project - which proposes a renewal of the arithmetic teaching in an algebraic perspective through a linguistic, socio-constructive and metacognitive approach. We discuss on its theoretical aspects and on our ways of working with the teachers in order to bring them to acquire ways of acting in the class in tune with the indications and aims of the project. We present a short excerpt of classroom discussion showing the appearing of first forms of algebraic thinking in the pupils and, more in general, we discuss about the difficulties detected in the teachers. We conclude with some remarks on the conditions which can produce a change of the teachers, on the difficulties to spread these innovations at large and on the one to control in the time their incidence on the development of a new view of algebra in the students.

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Keywords: Early Algebra, Algebraic Thinking, Teacher Education, Theory &Practice Relationship, Reflective Practice

ON EARLY ALGEBRA

Early algebra is a recently born disciplinary area, appeared in the syllabuses of leader countries such as UK and US and presently studied at international research level as far as the didactical transposition of its principles is concerned. Its origins can be traced back in the studies that, in the Eighties, were devoted to the difficulties in teaching and learning algebra. Its constitution as a disciplinary area is the result of studies that we briefly recall in the following lines.

As already known, the modern mathematics of the Sixties abandons the idea of mathematics as an abstract, static and isolated discipline in favour of a dynamic and evolutionary vision of it, rooted in the concrete world and open to interactions with different disciplines and contexts. The contents in the syllabuses are renewed and expanded, while a deep methodological revolution shifts the attention from the passive learning of mathematical facts towards problem solving and mathematical discovery.

According to these developments, there is a widespread awareness of the importance of investing on studies dedicated to problems of teaching and learning and to improve the culture of teachers affected by the novelties.

As far as the teaching of algebra is concerned, the first studies concentrate on the definition of projects of curricular innovation. The pioneers in this area are the English projects for the teaching of mathematics in grades 6th to 10th (e.g. Bell *et al*. 1985; Harper 1987), which promote an approach to algebra centred, from the very beginning, on modellization. Beside these projects appear studies of diagnostic kind, in search for the most widespread wrong behaviours in the pupils. Particularly important are Kieran's studies (1989), which clearly show how the problems in the learning of algebra are mainly due to the traditional teaching of arithmetic, completely regardless of relational aspects and of the control of the meanings implied by calculus processes.

In those years, scholars have different opinions on the relationship between arithmetic and algebra in teaching, as well as on the age at which pupils should be faced with algebra. As far as the first question is concerned, some scholars underline the epistemological rupture between the two areas (e.g. Filloy and Rojano 1989, Hescoviz and Lincevski 1994), others highlight the importance of looking at the two areas in a perspective of continuity, stressing their mutual synergies (e.g. Chevallard 1989-1990). As to the ideal onset age for algebra, some scholars maintain that this should occur in the 8th grade (e.g. Usiskin 1987), while other authors state that it should be tackled earlier, at primary level (e.g. Davis 1985). Nevertheless, there is general agreement on the need to set the foundations of the teaching of arithmetic in a relational and pre-algebraic perspective. The objectives are: a) to overcome classical difficulties and stereotypes such as the directional equal sign, or the lack of closure in arithmetic expressions; b) to induce a structural vision of arithmetic expressions, detecting equalities or analogies and differences; c) to open the way to generalization processes and modellization for the genesis of the objects of algebra.

The acknowledgement of the importance of these issues is highlighted at the WG on algebra at ICME VII (Laval, 1992) where the scholars get to envisage a specific area of teaching they call pre-algebra: as a bridge between arithmetic and algebra (Lincevski 1995). This denomination turns then soon into early algebra. Moreover, the activities of syntactic transformation are no longer seen as isolated, but rather in their relationship with activities of representation and interpretation, Bell (1996) speaks of 'essential algebraic cycle'. Other scholars (Arcavi 1994, Arzarello *et al.* 1993, Boero 2001) open to the metacognitive dimension, shifting the attention towards the control of the properties that legitimate the processes of syntactic transformation and the activation of anticipatory thinking, seen as the ability to foresee (without carrying out syntactic transformations) new possible forms of an expression, by checking its meanings with reference to a given aim, or to hypothesize formal writings to be reached in order to achieve specific results. With reference to generalisation, Mason maintains that pupils should be lead to conquer the double awareness of '*seeing the general in the particular'* and '*seeing the particular in the general*' and, most of all, become aware of the plurality of cases contained in a general statement. In this field, many studies are devoted to figural and numerical sequences (e.g. Stacey 1989).

In the second half of the Nineties there is a flourishing of studies on these aspects mainly targeted at pupils aged 11 to 13. Some of the studies stand out for their theorising of models for a conceptual development in algebra of a socio-constructive type, which highlights the influence of the classroom environment on teaching within the framework of an algebraic vision of algebra as a language (e.g. Da Rocha Falcão 1995, Meira 1996, Radford 2000). In the US area there is widespread agreement on the idea that primary school syllabuses should be re-arranged, according to the social needs of the 21st century, in the sense of early algebra (Kaput 1998). Since 2000, early algebra issues have raised an increasing interest within the international scientific community, as testified by the many interventions devoted to this theme at international conferences, as well as by the production of specific monographs (Cai *et al.* 2005, Kaput *et al.* 2008, Caj and Knut 2011). Many experimental studies of classroom implementation of early algebra tackle simultaneously the issue of teacher training (e.g. the reports by Carpenter *et al.,* Kaput and Blanton, Daugherty in Cick *et al.* 2001 or Carpenter *et al* 2003, Kaput and Blanton 2005).

Our research studies are located within this frame and develop within the *ArAl Project: paths in arithmetic to favour pre-algebraic thinking,* a project that, as we shall clear later, develops through two merged educational paths: the first one is devoted to the planning and implementation of class activities and to the analysis of students learning; the second one is aimed at a renewal of teacher professionalism through individual and shared reflective practices on the activated didactical processes. (Malara and Navarra 2003, Cusi *et al.* 2011, Malara and Navarra in press, Cusi and Malara 2015). Before concentrating on the ArAl project, we recall here some questions related to the socio-constructive teaching and to teachers education.

SOCIO-CONSTRUCTIVE TEACHING AND TEACHERS EDUCATION

The socio-constructive model of mathematics teaching sees the pupils as creators of their knowledge through activities of collective work on problematic situations that allow specific properties or mathematical concepts to arise. In this model, mathematical discussion has a central role, since it makes pupils express their ideas and debate on the different points of view that emerge in the class, conquering new knowledge while sharing their constructions and consolidating them by reflecting on their meaning and role. The coordination of a mathematical discussion requires in the teachers methodological skills that go beyond the mere disciplinary competence. First of all, teachers must foresee the development of classroom action; then, they should make hypotheses about pupils' conceptual constructs and on the possible didactical strategies to help them modify such constructs. From a social point of view, they must be able to create a good interaction environment, stimulating participation, mutual listening and facilitating the production of counter-arguments; they should avoid judgement and ask the class to validate the arguments; they should ask questions at metacognitive level so as to allow pupils to internalize the processes carried out.

In order to develop these skills, the consolidated key practice in research environment is based on the importance of teachers' critical reflection on classroom processes, particularly when these reflections are shared. To foster teachers' self-observation ability during classroom action, Mason (2002) suggests the constant practice of the '*discipline of noticing*' and he underlines that teachers have to become aware "*not simply of the fact of different ways of intervening, but of the fact of subtle sensitivities which guide or determine choices between types and timings of interventions*" (Mason 2008, p.49). Jaworski (2004) supports the efficacy of "*inquiry communities*" (mixed groups of teachers and university researchers) and she stresses that this dialogical model is particularly relevant to teacher professional development because it makes teachers research on their own practice and, consequently, develop a critical intelligence and an increasing awareness of the different aspects to be faced during the teaching processes. Recently Schoenfeld (2013) has underlined the importance of studying the interaction between teachers' knowledge, resources, goals, beliefs and orientations and their in-the-moment decision making. This position is consistent with the studies which discuss the crucial role played by an investigation *in situ* of the interrelationships between teacher knowledge, beliefs and emotions because it allows the teachers to become aware of the mismatch between their espoused beliefs and the beliefs that emerge from practice (e.g. Sowder 2007 or Thames and Van Zoest 2013).

In the last decade several research projects for teacher education are planned focused on a real teachers' involvement and their active collaboration with researchers but where the reference to the theory is considered fundamental (e.g. Potari 2013). The teachers who collaborate in these project are motivated to undertake inquiry-based practice, to give time to collaboration with researchers, and to engage seriously with reflective developmental practice. We are in tune with these scholars (e.g. Malara and Zan, 2008): we believe that suitable research projects and teacher education programs should be conceived with the aim of actively involving teachers in the field of practice and, in the same time, motivating them to approach theory to elicit useful tools to promote their own practice and to become more sensitive in perceiving and interpreting class processes. We also agree with Thames and Van Zoest (2013), who stress the need of developing studies aimed at the analysis of the effects of specific teacher actions on students' learning. Therefore, in our educational programs we work with and for teachers with the aim of making them become more sensitive and able to consciously observe and control their behaviour, also in facing contingent actions and taking sudden decisions, but above all in controlling the incidence of how they teach the basic theoretical elements of our approach to early algebra, and the feedback of theoretical aspects on the pupils' attitude and arguments.

THE ARAL PROJECT

Our interest in early algebra derives from a broad study we carried out in the early Nineties, concerning didactic innovation on Algebra in middle school. In the beginning we detected in several students a certain rigidity which we could overcome only in long term, this fact motivated us towards the opportunity of operating in pre-algebraic key at primary school level. That's how the *Progetto ArAl* was born (Malara and Navarra 2003).

Our perspective in the approach to early algebra is linguistic and metacognitive, based on the hypothesis that there is a strong analogy between the modalities of learning natural language and algebraic language. The specific hypothesis on which the ArAl Project is based is that the mental framework of algebraic thought should be built right from the earliest years of primary school in arithmetic realm (but not only in it) through a didactical contract based on the principle *first represent then solve* and the construction of an environment which might informally stimulate the autonomous processing of that we call *algebraic babbling,* i.e. the experimental and continuously redefined mastering of a new language, in which the rules may find their place just as gradually, trough the production, the analysis and the discussion of initially imprecise representations. Within a game of translation and interpretation between sentences in natural language and in formal language the pupils: approach the use of letters; activate the construction and the elaboration of the first algebraic expression; construct and solve equations in a naive way, reflecting on the underlying processes; interpret the meaning of formal sentences as to specific questions. This way, they become aware of the meaning of signs and symbols and of their role in the formal writings.

The project develops within the classes through the exploration of situations with socio-constructive methods (discussion, argumentation, verbalization). Basic mathematical elements are: the symmetry of the equality sign; the plurality of representations of a number; the reification of the properties of the arithmetical operations through the analysis of problem solving ways; the comparison of the value of numerical expressions without doing calculations; the algebraic modelling and the solutions of verbal problems with unknown data; the individuation and the representation of numerical regularities and of correspondence laws[1].

In the ArAl project, the image of early algebra is expressed through a set of key words and concepts that refer to arithmetic and algebra, but it defines its areas evolving from both disciplines towards a different and original identity. We can consider it a meta-discipline, concerning not much the objects, processes and properties of arithmetic and algebra, but rather the genesis of a unifying language between the two, i.e. a meta-language. In order to control the meta-disciplinary knowledge of early algebra, the teacher acquires the meaning of words and linguistic constructs which represent new conceptions of intertwining between arithmetic and algebra. Here we shall only recall some of

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[1] Examples of classroom activities and related didactical processes, can be found in the *ArAl Units*. The Units can be seen as models of teaching/learning socio-constructive processes related to a specific arithmetic/algebraic theme. They are structured in such a way as to make the teaching process transparent to the teachers: methodological choices, activated class dynamics, key elements of the process, extensions, potential behaviour of pupils and difficulties they may encounter, ecc, all essential elements for their free reproduction. Moreover there are several materials (video of classroom activities, articles for the curricolar planning, commented classroom transcripts, …) which can be dawnloaded from the Project website www.progettoaral.it .

these terms that turned out to be fundamental in order to generate in the pupils ways of seeing that are functional to the development of algebraic thinking.

Some theoretical key elements of the ArAl project

We present here some theoretical key elements in our approach to early algebra: the '=' sign. representing vs. solving, canonical form / non-canonical form of the number; translations between languages and the respect of the rules.

The equal sign and the duality process-product. The usual reading of 5+6=11 is ‘5 plus 6 *is* 11’: what’s on the left to the equal sign is seen as an *operation,* whereas on the right it is seen a *result*. The two sides of the equal sign are interpreted as *ontologically different entities*. But the algebraic meaning is different: it indicates the equivalence between two representations of the same quantities, i.e. between two entities that are *ontologically equal*. We usually introduce this alternative perspective by discussing with teachers the words of an 8-year-old pupil: “It is correct to say that 5 plus 6 makes 11, but you cannot say that 11 'makes' 5 plus 6; so, it is better to say that 5 plus 6 'is equal' to 11, because in this case the other way round is also true.”

Representing vs. solving A widespread belief among pupils, favoured by the traditional teaching of arithmetic, is that solving a problem means identifying its result; this perspective focuses the attention on the operations. In order to bring about a change of perspective, it is necessary to move from the cognitive to the metacognitive level. In Aral project the pupils are brought to not immediately search the result of a problem but they are slowly guided forwards the metacognitive level, shifting the attention on its structure and represents it through algebraic language. The following example highlights the difference between the tasks 'Solve' and 'Represent'. Situation*:* Eva has 13 Euros. She receives 9 more Euros and spends 6 Euros. (a). ‘classical’ arithmetical task (procedural view): How many Euros are left to Eva? (b). Task in pre-algebraic view: Represent the situation in the mathematical language so that others can find out how many Euros are left to Eva. Task (a) emphasizes the search for the product (16), whereas (b) concentrates on the process (13 +9-6), i.e. the representation of the relationships between the elements in play. This shift of perspective also favours the naming to the quantity to be determined: representations such as “Eva left Euros = 13+9-6” often are simplified by the pupils introducing a letter for representing the Euros left to Eva. This difference is connected to one of the most important aspects of the epistemological gap between arithmetic and algebra: while arithmetic implies an immediate search for solution, algebra delays it and begins with a formal transposition of the problem situation from the domain of natural language to a specific system of representation.

Canonical and non-canonical representation of a number. In order to promote the comprehension of these concepts we resort to the strategy of writing on the blackboard the name of a pupil and some information concerning him/her (e.g.: son / daughter of, friend / A, partner / a of bank, etc.). The class understands that those are different ways to *call* the pupil: one is his/her name, all the other definitions (*representations*) expand the knowledge on him/her by adding information that the first name does not give. The teacher leads pupils to understand that the situation is similar with numbers: A number, 12 for example, has its own name which we call *canonical form*, all other expressions quantitatively equivalent to 12 – such as 3×4, (2+2)×3, 36/3, 10+2, 3×2×2, … - are their *non canonical forms* and each of them has its *sense* with reference to the underlying process and to the context which determines it. The concept of canonical / non-canonical form has crucial implications for both pupils and teachers, also in order to reflect on the possible meanings attributed to the sign of equality. Moreover, it becomes a kind of "*semantic ferry*" towards generalization.

Transparent representations vs. opaque representations. A representation in mathematical language is made of symbols that convey meanings, the comprehension of which depends both on the representation in itself and on the ability of those who interpret it. For example: the non canonical form $2^2\times3^3\times5^2$ gives more information on the divisors of the number 1350 than its canonical form (1350). We can therefore talk of a higher *opacity* for writings such as 1350, of a higher *transparency* for those like $2^2\times3^3\times5^2$. Generally speaking, the transparency of the process favours the control of meanings, highlighting the underlying properties; it allows to understand possible errors and to clear up possible misconceptions which may arise.

Translations between languages and the respect of the rules. A key aspect is making students understand the importance of respecting the rules of algebraic language. While students start soon interiorizing the importance of respecting the natural language's rules in order to facilitate communication, it is difficult to make them develop a similar awareness in relation to algebraic language. It is therefore necessary to help them understand that algebraic language, too, is a finite set of arbitrary symbols which can be combined according to specific rules to be respected. In our project, this kind of conception is fostered through the creation of linguistic mediators which force the respect of rules in communicating even advanced concepts by means of algebraic language, in a perspective which foster generalization. A mediator that teachers particularly appreciate for its efficacy (both in conveying the construction of the algebraic language and to motivate the interpretation of formal writings with reference to given contexts) is Brioshi, a virtual Japanese student who doesn't speak the Italian language but knows how to express himself using a correct mathematical language. He is an algebraic pen friend with whom students communicate using mathematical sentences which should be written through a correct application of syntactical rules in order to be understandable.

TEACHERS EDUCATION: OUR APPROACH

At the basis of our project is the hypothesis that a *fruitful exchange between theory and praxis* can make teachers' competence evolve at two different levels: first, in *picking up signals* in everything that co-defines their professional setting, either *on the spot* or in the organization of their *theoretical tools*; secondly, in *processing the received signals* so as to convert them into their own cultural patrimony.

In order to facilitate the teacher's approach to theory, we have conceived a *Glossary* (to be found at www.progettoaral.it), presently made up of almost 150 interconnected keywords referring to five areas (general, linguistic, mathematical, socio-didactical, psychological). It is structured so as to avoid pre-defined approaches to the contained terms. The texts describing each term of the glossary show other keywords in bold type, so that each term puts the teacher into a double process of conceptual scrutiny and expansion: scrutinizing the term through the relationships of the key terms mentioned in its definition; expanding, since their bold print offers a potential incentive to further reading. The ArAl glossary represents a reference system that allows the teacher to gradually attain a global vision of early algebra, combining theory and practice by pursuing a linguistic conception of algebra, within which he/she should construct with the pupils an convincing control over its meanings.

As to the teachers' ability to interpret signals when they are in the classroom, improvement can be obtained through the increasing awareness with which they learn to transform the manifold occasional observations and reflections into the tool of a personal methodology, resulting from the interlace of observation skills, motivation to action and knowledge of *how* it would be appropriate to intervene. Our aim is that *teachers* approaching early algebra *become leaders of their own training*: willingness to take a challenge and reflect on their own action within sharing practices that gradually generate a new *forma mentis* in them, deeply different from the previous one. A key aspect of the training process we propose lies in leading the teachers towards the *ability to notice*, by showing them *on which aspects* they should concentrate and, simultaneously, helping them understand *how* to intervene.

With this aim, we involve teachers in an activity of critical analysis, and consequent reflections, of the transcripts of audio and video-recordings of classroom processes. It has the aim of highlighting the interrelation between the students' construction of knowledge and the teacher's behaviours in guiding them to perform this construction. The classroom transcripts are sent by teachers, together with their own comments, to e-tutors who make their own comments and send them back to the authors and other members of the team, who can add other comments. The transcripts so enriched become an interesting object for teachers education that we call multi-commented transcripts (MTs). Through the MTs, the teacher is guided towards re-reading his/her didactic interventions with 'theoretical glasses' on, referring both to the literature and to the ArAl Glossary. This process of analysis and reflection on the micro-decisions that the teacher makes during his/her lessons allows to objectify the limits and the problematic issues in his/her action, leading him/her to a conversion of his/her knowledge and beliefs on arithmetic teaching and algebra, towards a reconstruction of their personal attitudes in the classroom.

The comments in MTs highlight not only the positive aspects, but also the possible stereotypes, beliefs and behaviour that are often mistaken. Frequently, the comments underline that teachers need to have better control on using linguistic operative terms (*calculate, solve, find the result, it gives ...)* that inhibit the development not only of a relational vision in arithmetic, but also of the introduction of pre-algebraic terms (*connect*, *translate*, *represent*, *interpret*). Furthermore, suggestions are given on how to guide and help the pupils translate from natural language into the symbolic language. For instance: on dealing with the problem (presented to the classroom) of expressing arithmetically the term 'previous' – referred to a natural number – it sometimes happens that the pupils are only capable of linguistic paraphrases, such as 'that precedes', 'that comes before', without ever mentioning the number to which the term refers. This sometimes produces an empasse in the discussion and the teacher is led to give the solution to the question. Often, the comments – and even more the collective reflection – discuss on how teachers should guide the pupils to objectify the arithmetical connections between two natural subsequent numbers, so that they can get to express each of the two numbers through the other one. In situations like this, if one asks pupils to express the relationship between the numbers in different ways, one usually obtains the identification of the 'productive' point of view. Particular care is devoted to making teachers reflect on their action in front of 'false discussions' focusing on a dialogue between the teacher and some or few pupils, where the teacher continuously rhetorically suggests the answers, possibly through pleonastic questions such as: *is everything clear? Do you understand? Do you agree? Is it true? Is it ok for everyone?* without actually allowing the class to re-examine the whole discussion so as to check whether it has been effectively acquired. The general comments recommend that pupils be educated to argue thoroughly and coherently, with an appropriate use of language, underlining that the comprehension of mathematics occurs also through its collective and appropriate usage. Moreover, it is recommended that the teachers try to promote peer-dialogue interaction and limit his/her role as much as possible, to reflect on the importance of asking thought-provoking questions in the classroom and drawing back during the answer phase: if the teacher is the constant pivot of the discussion, the social aspects of knowledge construction are weakened.

In this scenery, the teachers learns again to manage the *socio-cognitive* processes, comparing its epistemology with the reference frames that they are offered and gradually developing the outcomes of the debate, so as to consolidate them as a steady cultural heritage. This process aims at making them grasp aspects (of their behaviour and in the pupils') that did not emerge at the beginning. Particularly effective is the collective debate on the classroom action of different teachers dealing with the same activity, since the action of one teacher can become a model of good practice for their colleagues.

The joint reflection on the MTs strongly influences the development of theoretical, methodological, instrumental, material aspects (Units of the ArAl Series, papers, articles, learning objects) and supporting elements (website, blog, Facebook Group) aimed at offering teachers a cultural background that can help them act differently in the classroom (see Figure 1. below).

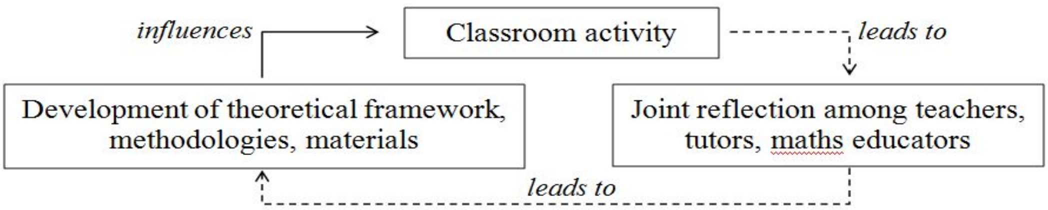
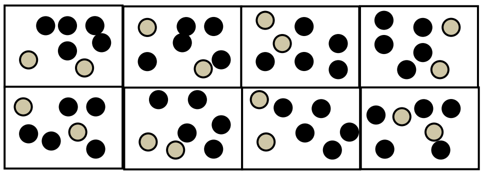


Figure 1. The cycle of teachers' mathematics education

Moreover the MTs allow to ascertain whether the classroom-leading strategies have changed (and *how*) during the training. Among the factors that influence the assessment of the teacher growth there are, for example, the following issues: Does the teacher develop a wide range of roles in order to promote a refection onto mathematical processes or objects? Does he/she foster linguistic interactions by encouraging verbalization, argumentation, and collective discussion? Does he/she negotiate and share with the pupils the theoretical framework? Does he/she modify their initial points of view or does he/she seem insensitive towards meaningful changes in his/her initial attitude? For reasons of space, here we cannot offer sufficiently meaningful examples of the insights achieved by the teachers and of the evolution they underwent through these practices; we therefore refer to Malara (2008), Cusi *et al.* (2011), Cusi and Malara (2015). We limit ourselves to a small example drawn from Malara and Navarra (in press), which highlights the pre-algebraic approach attained by the pupils.

Episode (pupils 8 years old, grade 3)



The pupils have represented this situation in the mathematical language and reflect on the suggestions made by Alice n=5+2×8, Martina 5+2×8=n and Ada n=(2+5)×8.

Francesco: I think they are right because 5+2 represents the marbles that are in a box, by 8 which is the number of the boxes.

Maria: They are the same as Ada's but they don't have parentheses.

Teacher: Let's reflect on the presence of the parentheses. Do they change anything?

Andrea: I think they do, because 5+2×8 is equal to 21, while (2+5)×8 is equal to 56.

Bruno: It's true: the teacher said once that in a chain of operations you solve multiplications first.

Maria: This means they are not the same!

Francesco: That's right, the translation with the parentheses is the more correct one.

There are many elements here that let us appreciate positively, at different levels, the teacher's action: (a) *mathematically*: she has introduced the pupils to the use of letters, to the priority of operations within expressions, to the use of parentheses; (b) *linguistically*: she has fostered the organization of meaningful, complete sentences; (c) *metalinguistically*: she has promoted the reflection on the mathematical writings and their comparison; (d) *socially*: by inviting pupils into discussion, she has let them interact without her influence, listening to each other and having spontaneous dialogues; (e) *methodologically*: she has shared the theoretical framework with the class, by spreading words such as 'represent' and 'translate'. On the other hand, we make the teacher notice and reflect on the fact that (2) Francesco has referred '5+2' *to the marbles* and not *to their number*; (2) Andrea's argument highlights a possible his diretional vision of the sign '=' focusing on the *result*, and pupils should be guided from the level of *calculations* to the level of *representations*.

CONCLUDING REMARKS

Implementing an aware and meaningful approach to early algebra implies that the teachers convert their beliefs not only on the subject itself, but also on its teaching. The socio-constructive approach requires didactical sensitivity, listening skills, polyphony management during the discussions and a great control of timing and classroom dynamics. Shifting the teachers' attention from results to processes and opening to relational activities and generalization in arithmetic require in the teachers a great revision of the teaching contents and a methodological renewal that allows pupils' group work and mathematical discussion in the class.

Thanks to the specific work methodology adopted with and for the teachers, some terms and theoretical constructs of our framework have proved themselves to be fundamental not only in the communication among teachers and between them and us, but also during classroom activities and in teacher-pupil communication. This represents a concrete example of the role played by theory in didactics of mathematics, when properly mediated: it contributes to re-founding a teacher's culture according to the current educational challenges and goals. Particularly, we refer to terms such as 'represent', 'translate', 'interpret' and their corresponding nouns with reference to the coordination of the verbal and algebraic registers; to terms such as 'canonical representation' and 'non canonical representation' with reference to numerical designations; to the attributes 'opaque and transparent' with reference to numerical representations concerning calculus processes; to the terms process / product with reference to the points of view arising in problem solving.

Our studies have shown a tight interconnection between the beliefs, attitudes and lexicon used by the teacher on one hand and the perspectives and language developed by the pupils on the other. When the teachers act in a conscious and effective way, the pupils' acquisitions become evident and increasing: from naive forms of algebraic babbling to a friendly attitude towards the use of letters for codifying and generalizing observed facts and, more generally, to the achievement of a vision and appreciation of mathematics as a constructive discipline. A critical aspect of the results concerns the possibility of observing in a more consistent way, and therefore – at least potentially – more significantly the component of continuity between primary and secondary school, in order to assess the relapses into an anticipated approach to algebraic thinking – or, in other words, of an approach to arithmetic in an algebraic key – on the evolution of the pupils' mathematical thinking.

Beside the successful aspects, the manifold analyses of the transcription of classroom processes have made us aware of the difficulties that the teachers meet on managing mathematical discussions, and how teachers need us to work beside them at a methodological and didactical level. The results obtained in more than a decade of study allow us to state that the refinement of the teachers' ability to observe and control themselves during classroom action cannot be achieved in the short term, since it develops slowly, provided that they interface constructively with colleagues, learning to constantly analyse the didactical processes in which they are involved and reflecting on the micro-decisions that determine their development and feeling of 'intellectual need' of referring to literature for support.

This can be hardly obtained extensively. From a social point of view, the problem is huge and would require strong investments. In my country, the situation is dramatic because of to the lack of qualified pre-service education and for the discontinuity of in-service training. One should, on one hand, create professional trainings / study paths that endow future teachers with mathematical knowledge for teaching, as indicated by Bass (2005), and at least biennial labs (thorugh triennial would be better) to train teachers to manage classroom action, to acquire higher skills in methodology and didactics, including an intelligent and critical usage of teaching tools (textbook, specific software, etc). On the other hand, universities should start long life training courses for in-service teachers, so as to delete the institutional rift between school and university: a difficult goal to accomplish, owing to the static nature of our academic system and the slowness at which the regional school councils – that should represent a link between the two worlds – work. Our belief is that, from a social point of view, it is urgent to invest resources for an adequate and widespread teachers' professionalism. Owing to the socio-political situation in my country, our hope is weak. Still, it is important to believe that somehow, sooner or later, things will change for the better.

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