GENERALIZATION PROCESSES IN THE TEACHING/LEARNING OF ALGEBRA: STUDENTS BEHAVIOURS AND TEACHER ROLE

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We give an overview of the literature on generalization with particular reference to the studies about the students' ways of thinking in the development of generalization in algebra. We discuss the teacher's role in guiding students to face algebraic generalizations and we report on our methods and tools to improve teachers'competence in teaching this kind of tasks in a socio-constructive perspective.

> To learn mathematics involves learning to think mathematically… The essence of thinking mathematically is recognition, appreciation, espression, and manipulation of generality. …
> The future of Arithmetics and Algebra teaching lies in teacher awareness of the fundamental mathematical thinking processes, most particularly, generalization. (J. Mason, 1996a)

1. THEORETICAL ASPECTS ON GENERALIZATION

A metacognitive teaching practice is necessary to give mathematics strength and meaning as a subject. In this type of teaching practice, the main tasks of the teacher are to lead the students to reflect upon how meaningful the procedures they choose are in front of the various situations, to make verbally explicit the strategies they implement, to compare them, to distinguish what is common and essential from what is not, to check the effectiveness of the representations they use. The aim of all this is to help the students focus on the unifying elements that emerge from the activity, getting to incorporate a variety of cases or situations in one single vision, to consider the strength of representations and to become aware of the process - object dynamics (Sfard 1991) which governs the reification of mathematical objects.

Basic elements of this type of teaching are generalization processes. By 'generalization process' we mean, briefly, a sequence of acts of thinking which lead a subject to *recognize*, by analyzing individual cases, the occurrence of common peculiar elements; to *shift attention* from individual cases to the totality of possible cases and *extend* to that totality the common features previously identified.

Detecting patterns, identifying similarities, linking analogous facts are all at the base of generalization processes; the key element in these processes is not the detection of similarities between cases, but rather the *shift of attention* from individual cases to all the possible ones, as well as the extension and adaptation of the model to any of them.

Generalization processes are natural: they emerge from our way of looking at things, of capturing them and of elaborating the products of our observations and experiences. They pervade human activities, although they are peculiar of the mathematical activity. Enriques (1942), writing as Giannini, discussing on the role of the error in the development of knowledge, writes:

> The path of the human mind is essentially inductive: that is to say, it goes from the real to the abstract. The understanding of the general should be conquered as a higher degree of something already known and easier, that is to say as a 'generalization'. On the other hand, the example has a clarifying property and, so, it is a strong instrument in scientific research and, at the same time, an invaluable tool for verifying and correcting theories. …The heuristic value of examples is even more evident, because everyone knows that the comparison between two different cases in which something in common appears is able to suggest to our mind the most beautiful generalizations and to show to us the best positions of problems …

It is also possible to generalize from the examination of one single case, when, regardless of its peculiar features, one sees it as representative of a whole area. The case is ‘exemplary’, i.e. it exemplifies the totality of cases. As in Hilbert’s renowned aphorism

> The art of doing mathematics is finding that special case that contains all the germs of generality.

Mason (1996a, 1996b) claims that ‘*generalization is the heartbeat of mathematics’* and that in the teaching of mathematics the students have to be brought to gain a double awareness: of ‘seeing the particular in the general’ and of ‘seeing the general through the particular’. As to the latter, he states the importance of the experience of ‘*examplehood’*, which brings the students to become aware of how a multitude of details can be subsumed under one generality. He writes (1996a, p. 21):

> One of the fundamental forms or experiences of a shift in the locus, focus, or structure of attention is the sense of ‘*examplehood*’: suddenly seeing something as ‘merely’ an example of some greater generality. To experience examplehood, in which what was previously disparate are now seen as examples of something more general, has an effect like cristallization or condensation (Freudenthal 1978[1], p. 272): it is releases energy and reduces the amount of attention required to deal with similar situations.

Mason underlines that the students’ recognition of a thing *as* an example requires that they grasp the sense of what the example expresses, the enhancement of the features which makes it ‘exemplar’ and the shading of the features which make it particular. Moreover he says that if the teacher is, at present, unaware of what makes exemplary the example, (s)he may not provide students with adequate support to appreciate the examplehood being offered.

Without disclaiming the efficacy of generalization as a didactical instrument, with reference to the inference of mathematical facts from the observation of few examples, Radford (1996a p. 107-109) poses the problem of the logical validity of the assumptions that come from that generalization[2]. He deplores the abuse of generalization in teaching, since the students may get the idea that the fact that a regularity occurs in few cases is enough to claim that it is valid as a ‘general rule’. It is therefore necessary to spend time working towards the recognition of the limitations of generalization, to distinguish between inductive and deductive processes and to become aware that the validity of an inductively inferred sentence can only be established through a proof.

However, it should be noticed that generalization processes in mathematics not only concern particular mathematical contents; they also involve meta-aspects, linked with the organization and structuring of the gradually acquired knowledge.

Harel & Tall (1991) reflect upon the modalities in which students, progressing in their studies, link together pieces of knowledge and enlarge their horizons. They detect how these moments of reorganization depend on the features of the students’ mental constructions and on the type of understanding (relational or instrumental) which underlies their knowledge. They distinguish between three types of generalization: 1) *expansive generalization* in which one extends his or her scheme without reconstructing it; 2) *reconstructive generalization* when a subject reconstructs an existing schema in order to widen its applicability range; 3) *disjunctive generalization* when, on

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[1] Freudenthal, H. (1978) *Weeding and Sowing: Preface to a Science of Mathematics Education*, Reidel, Dordrecht

[2] Radford introduces the issue by making reference to a renowned scene of ‘La cantatrice chauve’ (usually translated as The Bald Soprano) by Ionesco: at the Smiths’ they ring at the door, Mrs Smith opens but she doesn’t find onyone; the same happens at the second and third doorbell, at the fourth one she blurts with her husband, making a nonsensical inference, generalization of the prevoious cases « Do not send me to open the door! You have seen that it is useless! Experience has shown us that when we hear the doorbell,

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moving from a familiar context to a new one, the subject constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available.
They underline that expansive generalization is more frequent and easy to apply than reconstructive generalization, that the latter is delicate and subjective but also more effective, that disjunctive generalization is cognitively poor and turns out to be a real 'recipe for failure' for weak students: they are not able to see linking schemes and are helplessly submerged by the amount of notions.

Dörfler (1989, 1991) is interested in the modalities of construction of knowledge in the students, and he theorizes on the processes of generalization. He sees the generalization as a combination of cognitive processes at a double level: the subjective-psychological one, related to the individual-reflective dimension and the objective-epistemological one, related to the social dimension (sharing, communication and use of language). He considers knowledge as the result of the structuring and the organization of one's own experience and he views it as stemming from *appropriate actions* on certain objects through *reflection* upon both actions and transformations produced in the objects. In order to consolidate knowledge, he considers crucial the representation of a process '*by the use of perceiveable objects, like written signs, of the characteristic and stages, steps and outcomes of the actions'*. In this way a *protocol of actions* is generated which allows for a cognitive reconstruction and conceptualization of the process itself.

On these premises he develops a "*model of the processes of abstraction and generalization*" (Dörfler 1991). This model has its roots in Piaget's construct of 'reflective abstraction', a process where the actions are seen as genetic source of the (mathematical) concepts, but Dörfler enlarges the meaning of 'action' including also the symbolic actions. Two phases can be distinguished in the model: the first one, which leads to the emerging of invariants as well as the birth of representations for them; the second one, more meaningful from the mathematical point of view, where the focus is on the representations: through a reflection on them, the way of viewing them evolves and this leads to the reification of new mathematical objects.

More in details, the starting point of this model is an action or a system of actions (which are material, imagined or symbolic) upon certain (material or ideal) objects. In these actions one's attention is directed to some relations and connections between elements of actions. In many cases the actions combine the original elements in a certain and invariant way; when, repeating the actions (as often as one likes), the relations prove to be steady, these combinations and basic transformations emerge as "*invariants of actions*", defining the "*schema*" (of actions). Dörfler underlines: "*the emergence of the invariants* needs *a certain symbolic description*". This is a key point for the model. Symbols are used for the elements of actions or for quantities relevant for them, and for transformations or combinations on the objects induced by the actions. This representation of the invariants may include variable elements related to objects on which actions are carried out. The symbols (of verbal, iconic, geometric or algebraic nature) initially play a purely descriptive role: they represent either actions or transformations. This first phase can be summarized as one moment of *constructive abstraction*, where the original elements are substituted by *prototypes*, which better highlight properties or relationships we want to focus on (they gain meaning and 'existence' via the actions). The second phase develops through other two important moments:

- One moment of *extensional generalizations*, when the use of prototypes leads to determine the domain of variability of the patterns, which enhances the interchangeability of the objects with respect to the actions upon them. At this point the symbols lose their initial meaning of generic representatives and they acquire that of *variables with properties of substitution*.
- One moment of *intensional generalization*, when by reflecting upon the symbolic representations of the invariants, the used symbols lose their meaning of representatives (of variable elements of the actions), and they become elements of the action themselves and 'carriers' of the invariants: at this point symbols are detached from their range of reference and acquire a new meaning, intrinsically connected to the invariants, of *variables with the feature of objects*: so, a new mathematical object is born.

Dörfler claims that once a generality of this type is constructed, it becomes the basis for further generalization. He stresses that his model is a 'theoretical generalization' model, juxtaposed to 'empirical generalization' (EG), that is the Aristotelian basic process of finding a common quality or property among several objects or situations from sense perceptions. He states that EG does not contribute to the construction of the meaning of the concepts because it is mainly a recognition process, he criticises the use of EG in mathematics teaching and the fact that usually the ability to recognize the generality is postulated.[3]

Dörfler offers also an interesting sequence of examples of his model from both elementary and advanced mathematics. In these examples, however, the focus is uniquely on the mathematical contents, without specific reference to either the students or the teacher.

On the contrary Dörfler explicitly does not take into account the problem of what the appropriate starting situations for the students may be, and he devolves their choice to the teacher, since, he says, "*it is only she who knows the students and their interests*".

Later, Hejny (2003) proposes a model of construction and structuring of knowledge organized in six stages (see the table below) where generalization is viewed as a basic element, but still at a lower level than abstraction and functional to this. Hejny, referring to what Sierpinska[4] thought about the development of mathematical understanding, considers her vision as reductive, and he claims to agree with her, only if abstraction[5] is juxtaposed to generalization. An original element in Hejny's model is the fact that the student's motivation is seen as the first step of the process.

Comparing Hejny's model to Dörfler's one a first important difference can be noticed: Dörfler does not make a distinction between generalization and abstraction, he rather describes processes of generalization with moments of abstraction; on the contrary Hejny underlines that generalization abstraction is prior to abstraction.

The stages of development and structuring of knowledge in Hejny's model

1. *Motivation*. By motivation we mean a tension, which appears in a student's mind as a consequence of the contradiction between *I do not know* and *I would like to know.* This tension steers the student's interest towards a particular mathematical problem, situation, idea, concept, fact, scheme,...
2. *Stage of isolated (mental) models*. The acquisition of an initial set of experiences. At first, these experiences are stored as isolated events, or images. Later on, it might be expected that some linkage between them occurs.
3. *Stage of generalisation.* The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme that generalizes all these models is discovered. The process of generalisation does not change the level of the abstraction of thinking.
4. *Stage of universal (mental) model(s).* **A** general overview of the already existing isolated models develops. It gives the first insight into the community of models. At the same time, it is a tool for dealing with new, more demanding isolated models. If stage 2 is the collecting of new experiences, stages 3 and 4 mean organising this set into a structure. The role of such a generalising scheme is frequently played by one of the isolated models.
5. *Stage of abstraction.* The construction of a new, deeper and more abstract concept, process or scheme which brings a new insight into the piece of knowledge.
6. *Stage of abstract knowledge.* The new piece of knowledge is housed in the already existing cognitive

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3 As to this Dörfler considers the derivative concept and the 'examples', such as velocity, gradient, density, usually used to show the derivative as their common structure but- he stresses- this structure is not developed by the students themselves.

4 Sierpinska, A. 1994, *Understanding in mathematics*, London & NewYork: The Falmer press

5 Hejny (2003) writes: "In her analysis of the act of understanding, Sierpinska considers four basic mental operations: identification, discrimination, generalisation and synthesis. 'All four operations are important in any process of understanding. But in understanding mathematics, generalisation has a particular role to play. Isn't mathematics, above all, an art of generalisation? *L'art de donner le même nom à des choses différentes*, as Poincare used to say?' Sierpinska (1994, p. 59). We agree with this statement provided that '*donner*' covers both our terms generalisation and abstraction".

network, thus giving rise to new connections. Sometimes it ends up in the reorganisation of either the mathematical structure or a part of it.

A second difference concerns the role of representations. In Hejny's model the representation issue does not even appear, while for Dörfler it is essential, since the role played by symbols in the representation of invariants and the progressive change of meanings associated with them allows for the reification of mathematical objects. Another element of difference in the work of the two authors concerns the nature of the examples given for their model. While Dörfler presents examples focused on the mathematical content, with no reference to the subjects involved in the process, Hejny analytically shows the ongoing process of construction of knowledge through excerpts from the students' activities and dialogues which testify the moments when generalizations and abstractions are generated. In this sense, drawing on Sfard (2005)'s classification on the time periods that mark the evolution of mathematics education research, Dörfler's study can be placed in the 'content's era' whereas Hejny's research is fully placed in the student's era.

Regarding students, a broad and interesting piece of research is due to Ellis (2007), a teacher-researcher. The research object is the identification of students' key behaviours in the generation of generalizations. Ellis starts from the analysis of studies in mathematics education dealing with students' processes of generalization and she identifies three categories of actions that are typical of generalization: (a) the development of a rule that serves as a statement about relations or properties;

THINKING ACTIONS IN THE PRODUCTION OF GENERALIZATIONS (Ellis 2007)

ACTIONS OF GENERALIZATION

I *RELATING*

- *relating situations:* the formation of an association between two or more problems or situations. a) *connecting back* (the formation of a connection between a current situation and a previously-encountered situation); b) *creating new* (the invention of a new situation viewed as similar to an existing situation);
- *relating objects*: the formation of an association of similarity between two or more present objects. a) *property* (the association of objects by focusing on a similar property they share); b) *form* (the association of objects by focusing on their similar form)

II *SEARCHING*

- *searching for one same relationship*: the performance of a repeatead action in order to detect a stable relationship between two or more objects
- *searching for one same procedure:* the repeatead performance of a procedure in order to test whether it remains valid for all cases
- *searching for one same pattern:* the repeatead action to check whether a detected pattern remains stable across the cases
- *searching for the same solution or result:* the performance of a repeatead action in order to determine if the outcome of the action is identical every time

III *EXTENDING*

- *Expand the range of applicability*: the application of a phenomenon to a larger range of cases than that from which it originated
- *Removing details*: ther removal of some contextual details in order to develop a global case
- *Operating*: the act of operating upon an object in order to generate new cases
- *Continuing*: the act of repeating an existing pattern in order to generate new cases

FORMULATION OF GENERALIZATION

IV. *IDENTIFICATION OR STATEMENT*

- *continuing phenomenon*: the identification of a dynamic property extending beyond a specific instance;
- *sameness*: a). *common property*: the identification of a property that is common to objects or situations; b) *objects or representations*: the identification of objects as similar or identical; c) *situations*: the identification of situations as similar or identical);
- *general principle*: a statement of a general phenomenon. a) *rule*: the description of a general formula or fact; b) *pattern*: the description of a general pattern; c) *strategy or procedure*: the description of a method that can be extended beyond a specific case; d) *global rule*: the statement of the meaning of an object or idea).

V. *DEFINITION:* the definition of a class of objects all satisfying a given relationship, pattern, or other

phenomenon.
VI. *INFLUENCE:* a) *prior idea or strategy*: the implementation of a previous generalization); c) *modified idea or strategy* (the adaptation of a existing generalization to be applied to a new problem or situation).

(b) the extension or espansion of one's range of reasoning beyond the case or cases considered, and (c) the identification of commonalities across cases.

The scholar regrets that these studies essentially address the students' difficulties regarding the production of a law which is predetermined by the researchers, and that, consequently, the latter neglect to consider possible generalizations that are partial or not fitting with what is expected from the students[6]. She puts herself in a wider perspective and, in her observation of the students, she considers processes of generalizations as well as processes of transfer through which the students autonomously transfer and adapt their knowledge to new contexts, acting under different conditions.

Ellis investigates how students extend their reasoning, examines the sense given by students to their general claims, explores which types of common characters the students might perceive throughout the cases. The activities proposed to the students are various and very diversified and allow for the analysis of processes and outcomes. The wide range of the collected data (students' protocols, interviews, video transcripts of the class processes) allows her to develop a taxonomy on two macro levels: that of the *generalizing actions* and that of the *reflection generalizations.* (see the previous table).

Several other studies concern the processes of generalization in algebra which we refer to in the next section.

> What matters is how our eyes combine the images that have chosen to assent to be captured, how we are able to associate them playing back and forth, how we follow intuitions, alternative paths, possibilities […] (Davide Enia, *Palermo-India*, 2010)

2. GENERALIZATION AND THE TEACHING OF ALGEBRA

Processes of generalization are dominant in a teaching of algebra which gives room to generational and meta-activities in the sense of Kieran (1996). At the international level few studies address processes of generalization at an advanced level, on non standard problem solving activities (Papadopulos & Iatridou, 2010, Zazkis & Liljedal 2002). The majority of the studies concern processes of generalization in generational activities and are intertwined with the introduction of letters to encode the observed regularities in general terms. Kaput (1995) writes:

> "both the means and the goal of generalising is to establish some formal symbolic objects that are intended to represent what is generalized and render the generalization subject to furter reasoning"
> […] "acts of generalization and gradual formalization of the constructed generality must precede work formalism – otherwise the formalism have not source in student experience" .

Kaput is recognized as one of the fathers of early algebra, a disciplinary area which is now well established, which proposes the early use of letters intertwined with a relational teaching of elementary number theory as well as a valorization of algebraic language as an instrument to represent relations and properties, to carry out reasoning patterns and produce justifications. His studies gave birth to interesting experiments in the US which invested both the curriculum, by

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[6] Ellis writes: "studies examining students'generalizations often report students'difficulties in recognizing, using and creating general statement. Because work on generalization predetermines what types of knowledge counts as general, it may fail to capture instances in which students may perceive a common element across cases, extend an idea to incorporate a larger range of phenomena, or produce a general description of a phenomenon, regardeless of its correctness. … Focusing on correct mathematical strategies, mental acts that cut across strategies may be overlooked and generalizing processes that result in incomplete or incorrect generalizations may be omitted."

making students get closer to the generalization of facts, procedures and reasoning patterns, and teacher training (Kaput & Blanton 2001, Blanton & Kaput 2001, Carpenter & Levi 2001, Carpenter, & Al. 2003, Carraher & Al. 2000, 2001, Schliemann & Al. 2001). Influences of these studies can be found in the NCTM's proposals for the curriculum, where there is a strong emphasis on students' learning to make generalizations about patterns. Regarding this topic, the anticipatory studies carried out by Stacey (1989), Lee (1996), Orton & Orton (PME 1994, 1996) and the books by Mason & Al. (1985) and by Orton (1999) must be mentioned.

As a rule, international studies about the approach to algebra that involves the processes of generalization concern the study of: patterns, algebraically representable functional correspondences between pairs of variables, equations, structural aspects of arithmetic operations, simple numerical theorems (formulation of conjectures and their justification). However the study of patterns is the more practiced one, as it is also documented by the ZDM special issue *"From Patterns to generalization: development of algebraic thinking"* (2008).

Dörfler, in his comment to this issue, makes a few remarks we agree with (see Dörfler 2008). First of all, he claims that the knowledge and mastery of algebraic notations do not develop simply by generalizing patterns of various kinds. In particular, he observes that it is not enough for pupils to be able to translate expressions from the verbal to the algebraic register, if they are to grasp the meaning of formal expressions; he points out the importance of the "negotiation of the intended meaning of the algebraic terms, specially of their ascribed generality", because it is "the habit of usage of, of operating with, of talking about, etc, the marks/letters on paper" which makes the students aware of the meanings they bring. About the figural sequences he stresses the importance that the students become aware that a given visual cue can be seen in different ways and then look for its different views. Moreover, both to give room to the students' creativity and not to determine in them the stereotype of the existence of one 'unique law', given a series of figures, he suggests that it should be asked "*how can you continue*?" or "*what can you change and vary in the given figures*?"

Similarly, about the activities of modeling of functional relationships he states that "verbal or quasi-variable generalizations[7] will not easily permit one to even think of those properties of a functionl relationship. They describe the respective generality but they are not amenable to operate in it or with it". He also stresses that what makes productive the use of letters that allow to transfer the reasoning on the facts at stake into the calculations, is the chance to operate with the letters according to the common rules of arithmetic (condensed in the notions of ring or field); yet if the students are not aware of the possibility of actions, such as "adding" or "multiplying", on the letters, the sentence "*n stands for an arbitrary number*" remains void and difficult to be accepted.

Moreover, he claims that the papers presented in the ZDM issue do not clarify the relationship between this kind of activity and the mastery of algebraic calculations, which the students need to practice in order to become able to develop reasoning and produce proofs through algebraic language. Last but not least, he stresses that many papers are only focused on the difficulties met by the students, but that is reductive: the students behaviours and cognition can be influenced by the teacher's methods and ways of posing problems. On these aspects we shall come back later.

As to the literature, due to space reasons, we only take into account some among the most wide-ranging and consolidated studies, precisely those by: Cooper & Warren, Rivera & Rossi Becker and above all by Radford. Before dealing with them, we would like to mention a particular study by Ferrari (2006) about the generalization and formalization of solution processes for numerical problems in a primary school; here children are guided to make a distinction between data and numerical value of the data and are faced with the task to express the procedure followed to solve the problem in general terms. In this process the letters are adopted by the pupils as short names for a voluntarily not defined quantity of data to emphasize the expression of arithmetic relations among them; each expression is made according to the operational acts needed to solve the problem,

[7] This construct is defined later.

getting to represent the solution procedure in an algebraic expression. The results not only show the effectiveness of the approach: they also prove a strong involvement of the pupils which generates motivation to study the discipline.

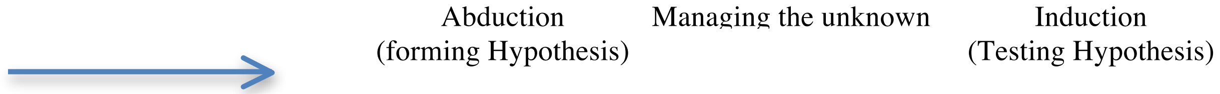
2.1 Cooper & Warren studies

The studies by Cooper & Warren (2008, 2011) concern the devlopment of an Early Algebra Thinking Project (EATP) aimed at placing early algebra activities in the Queesland Years 1-10 syllabus. They consider three main topics: a) patterns and functions; b) equivalence and equations; c) arithmetic generalization. The scholars, in the line of Radford, do not see algebra as the manipulation of letters but rather as a system charaterized by: indeterminacy of objects, analytic nature of thinking; symbolic ways of designating objects. Their obiective is the development of students' mental models based on relationships between real world instances, symbols, language, growth phenomena and graphs, particularly those that enable the modelling of real situations that contain unknowns and variables. In EATP they have studied the students' acts of generalization, in particular, pattern rules with growing patterns, change and inverse change rules with function machines and tables of values, balance principle in equivalence and equations, compensation principles in computations, abstract representation of change (e.g. tables, arrow diagrams, graphs) and relationship (equations), particularly looking at the relationships between representations and growth of algebraic thinking. These studies have reinforced their convinction that generalization is a major determiner of growth in algebraic thinking and preparation for later learning of studies. (Cooper & Warren, 2011). These authors, in analogy with the 'quasi variable' notion (Fuji & Stephens, 2001) - which espresses the students' recognition that a number sentence or group of number sentences can indicate an underlying mathematical relationship - introduce the *quasi-generalization* (QG) notion to indicate 'a step very near towards full generalization', i.e. the state where the students are able to express the generalisation in terms of specific numbers and can apply a generalisation to many numbers, and even to an example of 'any number', before they can provide a generalization in natural language and in algebraic notation. They have found that QG appears to be a needed precursor to the expression of the generalization in verbal or symbolic terms.

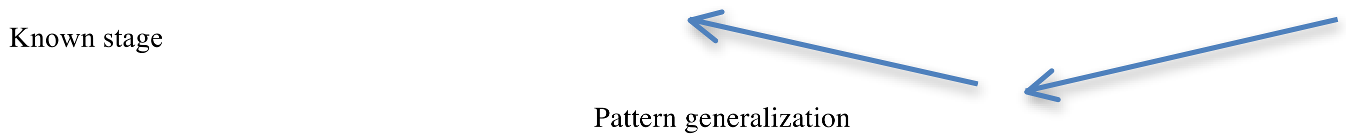
From the points of view of the classroom activities and of the students' side these studies are in tune with ours (see Cusi & Malara 2008, Cusi, Malara & Navarra 2011 and related references). But, as we shall show later, we take into account both the teachers' role in the class and, more in general, the issue of the development of their competence in leading the students to face algebraic generalization tasks.

2.2. Rivera and Rossi Becker's studies

The studies by Rivera (2010) and Rivera & Rossi Becker (2007, 2008, 2011) focus on the mental processes enacted by junior high school students to grasp and express linear (or quadratic) rules generated by the analysis of (figural stages of) non elementary patterns. The authors are interested in the students' construction and justification processes of their own generalizations. They focus their attention on the 'visual perception' as the result of sensory perception combined with cognitive perception, meaning, as far as the latter is concerned, the capacity of the individual to recognize a fact or a property as related to an object. They claim, like Radford, that the processes of exploration of a pattern are abductive-inductive, but differently from Radford[8], they incorporate in their model trial and error processes, accepting that cycles of abduction-induction may be repeated to refine the initial hypotheses, up to the definition of a rule which is suitable to generalization. The model produced for this process is the triangle indicated below.



8 We present later the Radford model



In particular, Rivera (2010) investigates in an analytical way the evolution of students' cognitive visualization, at the basis of the produced algebraic modeling. Concerning this latter point he refers to: Giaquinto (2007)[9] who maintains that the detection of the structure of a pattern arises from the association due to the natural 'visual power' of each one and from the use of a 'visual or perceptible template' which directs the exploration aimed at the recognition of either constant or redundant parts of a pattern; Davis (1993)[10] who conceives the "eye" as a "legitimate organ of discovery and inference" and who considers the discovery not only as the result of a logical reasoning path but also of noticing; Arcavi (2003)[11] who sees a visual template as a strategy to allow the students to see the unseen of an abstract world, dominated by relationships and conceptual structures not always evident; Metzger (2006)[12] for the "law of good gestalt" or "gestalt effect" concerning one's ability to perceive, discern and organize a figure. The author uses the expressions "patterns high (or low) in gestalt goodness" to express their high or low effectiveness to highlight the structure of a sequence. He shows the existence and effectiveness of visual templates in dealing with patterns which have linear or simple quadratic structures but he states that further research is needed in order to ascertain the possibility of visual templates in all figural patterns which have a not linear structure[13].

In Rivera & Rossi Becker (2011) the authors classify the procedures used by the pupils to reach an algebraic model of the sequence in three categories: 1) *Constructive standard generalizations* (CSGs); 2) *Constructive non standard generalizations* (CNGs); 3) *Deconstructive generalizations* (DGs). *The constructive generalizations* refer to those polynomial formulas that learners directly construct from the known stages of a figural pattern as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalt or parts. The *Deconstructive generalizations* refer to those polynomial formulas that learners construct from the known stages as a result of cognitively perceiving figures that structurally consist of overlapping parts (in some cases also embedding the pattern in a larger configuration that has a well known or easier structure).

The deconstructive ways of seeing a pattern imply that some elements (sides or vertices) of a figure can be counted two or more times and therefore the correspondent formulas involve a combined addition-subtraction process where overlapping elements have to be subtracted from the total. The terms "*standard*" and "*non standard*" refer to the algebraic expression of the rule: applying respectively if it is already simplified or not. From their studies CSGs appear to be dominant with respect to the DGs ones. The authors, even if they identify in the students' work the 'factual', 'contextual' and 'symbolic' Radford steps (see later), focus their analysis on the evolution of the students'work from figurally to numerically-driven (de)constructions. They document four types of justifications to support the formulas produced: extension generation; generic example use, formula projection, formula appareance match. They link the student' success with the classroom socio-cultural mediation which allows them to engage in multiplicative thinking and, in some cases, to simplify their justifications.

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[9] Giaquinto, M., 2003, *Visual thinking in mathematics*, Oxford University press

[10] Davis, R., 1003, Visual theorems, *Educational Studies in Mathematics*, 24, 333-344

[11] Arcavi, a. (2003), The role of visual representation in learning of mathematics, *Educational Studies in Mathematics*, 52, 215-241

[12] Metzger, W. (2006). *Laws of seeing*, Cambridge, MIT press

[13] Rivera realizes also a refinement of the previous model considering the starting triangle <abduction, induction, generalization> as a common base of two opposite tetrahedrons, where the top vertex represents 'the gestalt effect' and the bottom vertex 'the knowledge/action effect'. He considers a new research question, i.e. how this new model can be used in other algebra tasks involving generalization.

From these results the students' difficulty to produce CNGs is hardly understandable: since CNGs reflect faithfully the students'cognitive visions, in our opinion they should precede GSGs. Probably this behaviour shown by the students, depends on a clause of the didactical contract.

2.3. Radford studies

Radford develops a very refined set of studies (Radford 2003, 2006, 2008, 2009, 2010, 2011) where the ways in which 12-14 years old students immersed in a socio-constructive teaching, generalize linear patterns, are analyzed and theorized. We recall here some key points of Radford's theory.

The author claims that generalization implies two main processes which involve phenomenological and semiotic aspects: *grasping a generality,* a phenomenological act enacted through noticing how a local commonality holds across the given terms[14]; and *expressing a generality*, a semeiotic act enacted through gestures, language and algebraic symbols.

Grasping is seen as the enactment of an abduction from noticing some cases, i.e. the identification of a commonality meant as '*general prediction'* in the sense of Peirce. The abdution becomes a hypothesis through which, if positively verified, a new object emerges: a '*genus'*, i.e. a general concept arising by generalisation of the noticed commonality to all the terms of the sequence. An algebraic generalization occurs when the genus crystallizes itself into a *schema*, i.e. a rule providing one with an expression of whatever term of the sequence. This is Radford's model of this process (Radford 2008, p. 85)

Particulars

$P_1, p_2, \ldots, p_k$ Noticing commonality *C* Making *C* a hypotesis Producing the expression of p_n

abduction → Trasforming the abduction → Deducing p_n from *C* →

Later he argues that "the identification of the genus cannot be considered the result of an algebraic process " (Radford 2011). He claims that thinking development occurs both at the mental and the social plane, generated by material (gestures, language, and perception) and immaterial (imagery, inner speech…) components, which altogether constitute its '*semiotic texture'*. He considers that algebraic thinking is characterized by indeterminacy and analyticity which can be distinguished by the signs on which the student draw. As to the emergence of algebraic thinking he claims: a) that expressing generality algebrically does not imply necessarily the use of the letters (they can be used without any general meaning), instead of the *way of reasoning* which is made explicit in grasping and expressing vagueness in some way. b) the emergence of the algebraic thinking occurs when the students succeed to shift their attention from calculating a number of certain elements to the "*way of calculating*" such number.

Noticing students' behavior he distinguishes three levels of approach to generality. One first level, which he defines as 'naive induction', where there is no actual, aware generalization. It is characterized by pupils' trial and error processes, by the possible occasional discovery of generalities, by germs of abduction which are falsified in the checking stage. At this level, even though a rule may be expressed in the alphanumeric system the generalization is not algebraic. One second level, that he calls *arithmetic generalization*, where generalization is seen locally, in a recursive way, and expressed in the different cases through the addition of a constant term. One

[14] Dorfler (2008) critically reflects on the conception of 'grasping a commonality' as based only on an empiricist understanding. He considers the notion of circle and he underlines that it does not fit in with this vision because "nothing osservable have (exactly) the form of circle … in many situations the empirical generalization or abstractions need a complementary support by epistemic processes like idealization and hypostatization".

third level, which he defines as *algebraic generalization*, is a very mazy and complex one, marked by gradually more and more advanced phases. Regarding this latter level the author talks about a whole working area called *zone of the emergence of algebraic generalization*, which develops through ‘*layers of generality’*. The first layer, defined ‘*factual*’, is the one where the generalization appears by means of *concrete actions* on the examined cases, but it is not coagulated in a statement. The second layer, defined *contextual*, is reached when indetermination enters the discourse, pupils talk about the 'number of a figure' but they make space-time remarks on it, in a general perspective and a rule is expressed in various ways drawing on words, gestures, rhythms and signs. The level of the algebraic generalization is reached when pupils detach themselves from the figural context and shift towards the relations between constant and variable elements (numbers and letters). Important elements which intervene in this last process are *iconicity*, i.e. a manner of noticing similar traits in previous procedures, the shifting from a particular unspecified number to the level of variables *summarizing* of all the local mathematical experiences, the *contraction* of expressions which testifies a deeper level of consciousness. This is a synthetic representation of the processes (Radford 2006 p.15)

Radford’s model of the students’ strategies in dealing with pattern activities

Naïve Induction	**Generalization**			
Guessing	Arithmetic	Algebraic		
(Trial and Error)	(local recursion)	Factual	Contextual	Symbolic

In the most recent works by Radford (2010, 2011) the author adresses his attention to very young students (7-8 years old) and he studies in details the relationship teacher-pupils in classroom processes where the pupils are brought to detect and express generalizations in the exploration of figural sequences. In (Radford 2010) the scholar claims that “learning can be theorized as those processes through which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and action”. In particular he focuses on the ‘*way of seeing’* and states that “*the mathematicians’ eyes have undergone a lengthy process of domestication*” in the course of which people come to see and recognize things according to “efficient” cultural means.

Radford considers “seeing” not a simply physiological act but as a fruit of the cultural milieu where one is imbedded; he stresses that “*generalization rests on synthesizing* resemblances *between* different *things and also* differences *between* resembling *things”,* and that this game of visions has to be conveniently educated by the teacher. He highlights the social character of the teaching-learning processes, the role assumed by the teachers in it and focuses on “the way in which teachers create the possibility for students to perceive things in certain ways and encounter a cultural mode of generalizing”; he claims that *“*perceiving sequences in certain efficient cultural ways entails a transformation of the eye into a sophisticated theoretician organ”.

In the analysis of classroom transcripts he highlights the teacher’s behaviours (questions, guided reflections, gestures, tone of the voice, silences, looks) through which she succeeds to address her little students to become aware by themselves of the incorrectness of their visions and to autonomously correct them. As to this, he writes:

> …Poësis is a creative moment of disclosure – the event of the thing in consciousness … The poetic moment of disclosure of the general structure behind the sequence discussed in this paper was the result of a joint student-teacher interaction. This moment – the event of the thing in consciousness – was much more than a negotiation of meanings and an exchange. It was rather a Bakhtinian heteroglossic merging of voices, pointing gestures, perceptions, and perspectives … . (Radford 2010 p. 3)

From the examination of the studies we have considered, clear common elements appear about the articulation of the phases through which generalizations emerge, but there are also some elements of difference, for instance the different position of the trials in the models by Radford and Rivera about the students' behaviours in front of the exploration of figural sequences. Radford's studies stand out for the sharp intertwining between aspects of practice and theoretical aspects, and moreover for the consideration of the socio-cultural and epistemological dimension of both mathematics and its teaching. The experimental studies do not give esplicit indications about factors which contribute to the students'construction of the semantic basis for generalization. A study devoted to this aspect and carried out by my collaborators Cusi & Navarra, is presented in this conference.

In most of the studies we know, the teacher's role remains in the shadow. Warren (2006) states that more research needs to individuate teachers' actions and ways to pose questions which can facilitate the students' generalizations and Radford (2010, 2011) highlights the teacher's actions in guiding the students to 'see' analogies and differences among various stages of a pattern, but they do not mention that the majority of the teachers meet big difficulties to manage this type of teaching even when (s)he is convinced that it is appropriate to practice it. This problem has been an object of our studies and it is discussed in the next section.

3. OUR STUDIES ON THE SIDE OF THE TEACHERS' EDUCATION FOR A SOCIO-CONSTRUCTIVE APPROACH TO EARLY ALGEBRA

Since the nineties we have addressed questions of the teaching-learning of algebra and we have set up several experimentations of didactical innovation in collaborations with expert teachers - researchers. Our aims were to individuate the conditions of real applicability in the schools of didactical innovations in algebra, centered on algebraic problem solving, generalization, modeling and proof in the frame of a socio-constructive teaching. Our several studies have given birth to *ArAl Project: arithmetic pathways to favour pre-algebraic thinking*[15] (Malara & Navarra 2003) which proposes a revision of the arithmetic teaching in a relational key and an approach to early algebra of a linguistic-constructive type. The project involves students and teachers from kindergarten to the first biennium of upper secondary school but it is mainly devoted to primary and lower secondary school in a perspective of continuity between the two school levels.

The ArAl project is based on the hypothesis that there is a strong analogy between modalities in which natural language and algebraic language are learned. As we know, a child learns natural language through a large variety of situations which he experiences with an experimental attitude, gradually mastering the meanings and supporting rules of the language, up to the school age, when (s)he will learn to read and reflect on grammatical and syntactic aspects of the language. Similarly, the mental models of algebraic thinking and language should already be constructed in an arithmetical environment, even from the very first years of primary school, bringing a child to face pre-algebraic experiences in the arithmetical realm (grasping regularities, generalizing and expressing relationships, giving and comparing representations, extending properties by analogy, …). In this way (s)he can progressively develop algebraic thinking, in a strict intertwining with arithmetic, exerting a continual reflection on the meanings of the introduced symbols and of the implemented processes in classroom work.

As reported in Cusi & Al. (2011), our perspective of work in the classes is based on the following principles:

- The *anticipation of generational pre-algebraic activities* at the beginning of primary school, and even before that, at kindergarten, to favour the genesis of the algebraic language, viewed as a generalizing language. From these activities the pupil is guided to reflect upon natural language;

[15] ArAl is an acronym for « Arithmetic and Algebra ». The ArAl Project is led in collaboration with Giancarlo Navarra, a teacher-researcher who co-ordinates the organizational aspects of the Project and contributes to its scientific program.

it is from the analogy between the modalities of development of the two languages that the theoretical construct of *algebraic babbling* comes out[16].

- The *social construction of knowledge*, i.e. the shared construction of new meanings, negotiated on the basis of the shared cultural instruments available at the moment to both pupils and teacher. Arithmetic and algebraic knowings are both central, but they need to emerge and strengthen themselves through the coordinated set of individual competencies, which are the main resource on which they are constructed.
- The *central role of natural language* as the main didactical mediator for the slow construction of syntactic and semantic aspects of algebraic language. Verbalization, argumentation, discussion, exchange, favour both the understanding and the critical review of ideas. At the same time, through the enactment of processes of translation, natural language sets up the bases for both producing and interpreting representations written in algebraic language. From this centre, attention is then extended to the plurality of languages used by mathematics (iconic, graphical, arrow-like, set-theory language, and so on).
- *Identifying and making explicit algebraic thinking, often 'hidden' in concepts and representations in arithmetic.* The genesis of the generalizing language can be located at this 'unveiling', when the pupil starts to describe a sentence like 4×2+1=9 no longer (not only) as the result of a procedural reading 'I multiply 4 times 2, add 1 and get 9', but rather as the result of a relational reading such as 'The sum between the product of 4 times 2 and 1 equals 9'; i.e. when he/she talks about mathematical language through natural language and does not focus on numbers, but rather on relations, that is on the *structure* of the sentence.

In an approach of this type *the teacher has a key role*. In fact (s)he needs to set up a teaching strategy that allows for the implementation of an authentic socially shared mathematical activity, where space is given to linguistic aspects, to the representation of information and processes, as well as to meta-cognitive aspects. The latter are important to monitor the appropriateness and suitability of representations, to recognize and identify equivalent ones and select the best ones. All this requires a deep restructuring of the teachers' conceptions about both the contents to be taught and the teaching methodology in the classroom: a real 'culture of change' is entailed.

For reshaping teachers' professionalism several scholars stress the importance of a *critical reflection* by teachers on their own activity in the classroom (Mason 1998, 2002, 2008; Jaworski 1998, 2003, Lerman 2001, Shoenfeld 1998). Mason, in particular, proposes the study of the *discipline of noticing*. He claims that the skill of consciously grasping things comes from constant practice, going beyond what happens in the classroom, and recommends the creation of suitable social practices in which teachers might talk-about and share their experience. Also Jaworski stresses the effectiveness of *communities of inquiry*, constituted by teachers and researchers, emphasizing how teachers' participation in these groups helps them develop their individual identity through reflective inquiry. Our teacher education model follows these conceptions and modalities. But it represents the outcome of research and training practices developed in Italian Universities since the 1970's.

Instruments, methods and activities outlined and tuned in the ArAl project, work as a support for teachers to propose early algebra activities in the classroom, using a socio-constructive methodology, and, at the same time, as a training to become *metacognitive teachers* through a reflection upon their own action in the classroom. Follow-ups of the basic activities are twofold:

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[16] We call algebraic babbling the experimental and continuously redefined mastering of a new language, in which the rules may find their place just as gradually, within a teaching situation which is tolerant of initial, syntactically "shaky" moments, and which stimulates a sensitive awareness of formal aspects of the mathematical language. We employ the "babbling" image because when a child learns a language, (s)he masters the meanings of words and their supporting rules little by little, developing her/his knowledge gradually by imitation and self-correction or with the adults' support.

- on the pupils' side: the aim is to analyze the conditions under which pupils, since grades 4-6 manage, at a first level, to generalize, formulate properties and produce formal representations and, at a second level, to appropriate the meaning of algebraic expressions and become aware of their expressive strength;
- on the teachers' side: the aims are on two levels as well. One aim is to refine their ability to guide the class in the approach to early algebra following these ArAl modalities; a second aim is to foster their professional development through stimuli deriving from participation in at least two-year collaboration projects, characterized by the immersion in a community of enquiry on one's own practice, in a continuous interplay of *reflection*, *exchange*, *sharing*.

Our hypothesis for the promotion of the teachers' professional development is to bring them to be embedded in an 'environment' where they can acquire a new way to operate in and for the class, work actively and reshape their professionalism through frequent exchanges of studies, experiences and reflections. Our modalities of work in teachers' education are aimed at both bringing the teachers to analyze their didactical processes to assess their results and guiding them to reflect on these processes according to three different points of view: the development of the mathematical construction; the teacher's actions; the participation of each individual in the collective construction of the knowledge.

We believe that by observing and critically reflecting on socio-constructive teaching/learning processes, the teachers are led to become aware of the different roles they are supposed to play in the classroom, of the best ways to interpret them and can also get useful suggestions about how to behave in the classroom. Moreover, we believe it is crucial for teachers to be familiar with research results that can be useful for practice and to become aware of the importance of studying them for their own professional development.

The teachers who choose to participate in ArAl teaching experiments are mainly motivated by their 'first encounter' with the project through publications, congresses or events in the schools. Often these teachers have already studied the project and in particular its units[17] and the glossary that can be found in the project's website[18]. When they actually face the teaching experiment, they nevertheless show uncertainty towards class discussions, felt as open and unpredictable situations, difficult to be managed.

Through our studies we became aware of the difficulties that the teachers meet both in planning and in guiding classroom mathematical discussions. Our studies highlighted how during a classroom discussion often the teachers assume not adequate behaviours or fall back to a trasmissive teaching model. Therefore often they do not share with the students the goals of a problem exploration, they do not give room to some potentially productive interventions, they tend to ratify immediately the validity of some meaningful contribution without giving the class any opportunity to validate them. An example of a discussion where the teacher has this kind of behaviour is reported in appendix with a comment.

As a support to teachers and an answer to their needs, a mentor-researcher is associated with each group of teachers involved in the same teaching experiment: teachers and mentor share some moments of work face to face together with a dialogical relationship via e-mail. There are also regular working sessions of small groups with their mentor and the researcher in charge of the

[17] The units can be viewed as models of sequences of didactical projects, open to the teacher's choices and focused on a specific strand of activities. They provide information on the mathematical meaning and the objectives of the single activities presented, report excerpts that exemplify class discussions, as well as comments on both pupils' behaviours (meaningful constructions, frequent attitudes, difficulties) and on teachers' behaviour (appropriate interventions, ways of introducing and managing issues, attitudes etc.).

[18] The units are supported by the theoretical framework and, most of all, by the glossary, available online on the project's website <www.aralweb.unimore.it>, where teachers can find clarifications and further material on mathematical, linguistic, psychological, socio-pedagogical and methodological-didactical issues and also find prototype didactical sequences, aimed at giving them a stimulus for their-own elaboration of the highlighted teaching processes.

group, but also collective sessions, involving all the researchers and teachers experimenters, all held in schools or at the university.

Believing that observation and critical-reflective study of classroom-based processes help teachers become aware of the processes involved in every discussion and of the variables that determine those processes, our objective is to lead the involved teachers: a) to become increasingly able to interpret the complexity of class processes through the analysis of the inner *micro-situations*, to reflect upon the effectiveness of their own role and become aware of the effects of their own *micro-decisions*; b) to be in a better and finer control of both behaviours and communication styles they use; c) to notice, during classroom activity, the impact of the critical-reflective study undertaken on pupils' behaviour and learning.

In order to achieve this objective, we involve teachers in a complex activity of critical analysis of the transcripts concerning class processes and of reflection upon them, aimed at highlighting the interrelations between knowledge constructed by the students and behaviour of the teacher in guiding the students in those constructions. The analysis is carried out by building up what we call '*Multi-commented transcripts* (MT), or 'the diaries'. They are realized after transcribing in a digitally formatted text the audio recordings[19] of lessons on topics that were previously agreed with the researchers. They are completed by the teachers-experimenters who send them, together with their own comments and reflections, to mentors-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers. Often the authors make further interventions in this cycle, making comments upon comments or inserting new ones. This methodology is characterized by a sort of web choral participation, due to the intensive exchanges via e-mail which contribute to the construction of the MTs, and to the fruitfulness of the reflections emerging from the different comments.

Here we only propose a short excerpt from an MT, trying to show how this instrument enables one to highlight the behaviours enacted by teachers, the difficulties they meet and the awareness they achieve after the work of analysis and reflection has been carried out on the basis of the received comments. We are well aware that this excerpt cannot fully express the richness and the variety of the questions which arise from the classroom transcripts, the type of interactions with the teachers that the comments allow and how these can help them to refine their actions in the class, so we refer to other examples which can be found in Malara (2008), Malara & Navarra (2011), Cusi & A. (2011) Cusi and Malara (in press). In order to preserve the discussion flow, analytical comments are reported in the same order in which they were made. Authors of comments are labelled as: T: teacher; M: mentor; R1-R2: team researchers.

A short example of MCT

The teacher proposes a topic concerning the exploration of a sequence, given the first three terms (it is the arithmetic progression with initial element 4 and step 7). The activity is aimed at determining a general representation of the sequence. In the following excerpt, the class (grade 6) had already identified the sequence's recursive generating law. The teacher writes the following table on the blackboard and opens up a discussion to introduce the class to the study of a representation for the general correspondence law (T represents the teacher; S, J and A represent the students involved in this part of the discussion).

[19] We chose to analyze audio-recordings instead of videos of classroom processes because we believe (Malara & Zan 2008) that, while watching the video may not enable teachers to completely capture the details of the verbal interaction, analyzing transcripts, instead, fosters the crystallization of interactive processes and highlights gaps, crucial decision making moments and also omissions, oversights, carelessness.

Sequence ranking number	Sequence number	Operations made to jump from the place number	'Mathematical recipe[20]' to construct the number
1	4	4	
2	11	4 +…	
3	18	4 + … + …	
4	25		
5	32		

1 T: How do we get to 11?
2 S: + 7.
3 T: We make 4 + 7. What about the third place, S? We make…?
4 S: 4 + 7 + 7.
5 J: Wouldn't it be better to make 4 × 2? **(1)**
6 T: What about the fourth place?
7 S: 4 + 7 + 7 + 7.
8 T: What about the fifth?
9 S: 4 + 7 + 7 + 7 + 7.
10 T: What if we had a sixth place?
11 S: 4 + 7 + 7 + 7 + 7 + 7.
12 T: Correct. So, now we find…
13 A: I didn't get it. What do I put in the first place?
14 T: Well, there is 4 in the first place.
15 A: I put 4 × 1. **(2)**
16 T: Well, but there is no '×' there. The first place is 4 **(3)**

Comments

(1) M. Why doesn't T comment upon J's intervention?
R2. I agree. Probably J grasps a regularity but doesn't express it correctly, instead of saying 4 + 7 × 2 he packs everything in 4 × 2. T should have clarified this.

(2) R1. Also this intervention might have been investigated. Why does A think about the product of 4 and 1?
R2. Again we are in front of a badly expressed intuition. The student probably wants to 'fill the gap' he sees in the representation of the first term as compared to the others. Here T misses the chance to change the representation of the first term, 4, into one that fits with the situation, for example writing 4 as 4+0 and getting back to the class posing the problem to find a representation for the first term, similar to the other ones.

(3) R2. This intervention by T suggests that she excludes the possibility of representing 4 in another way, thus showing little algebraic farsightedness. It would be extremely appropriate to encourage these intuitions, although imprecise, trying to redirect them.
T. All these remarks make me think I am really close-minded and I didn't realise it before. I don't know whether this is a matter of attention, of being used to seeing things in different ways, of fearing to get out of the scheme to be followed (or the one I thought I should follow).

Analysis of the excerpt

This excerpt documents a number of rigid behaviours by the teacher in her action. She does not manage to productively value the intuitions of some pupils, blocking their emerging mathematical explorations (lines 5, 13, 15) and to direct pupils towards a relational reading of the correspondence, which implies the use of the multiplicative representation (line 16). If we look at

[20] The expression "mathematical recipe" is a metaphor used by the teacher to convey the idea that pupils should use a representation of the sequence's number in function of the place number.

the comments she proposes, we notice that she only makes remarks about her action in the class after reading both mentor's and researchers' remarks. Her *a posteriori* comment shows awareness of her own rigidity and of her tacit fears to leave usual schemes to approach innovative activities (note 3-T).

Comments made in this excerpt reflect some of the categories we already highlighted (Malara 2008) and that seem to be strictly interconnected here: (1) conceptions linked to cultural and/or general educational issues (note 3-R2); (2) methodological issues concerning mathematical aspects (notes 1-R2; 2-R2; 3-R2); (3) management of discussions in the classroom (notes 1-M; 2-R1). Further categories strongly emerged in MCTs- not documented here for space reasons- refer to the distance between theory and practice (difficulty in drawing on elements of the theoretical framework) and to a wide range of linguistic issues.

The example we presented shows the role of MCTs in the training program in which teachers are involved, reminding that this analytical work is carried out on the transcripts of all the episodes that constitute the teaching-experiment. It is through the comments that teachers: (1) actually realize how the development of pupils' mathematical constructions is strongly affected by the teacher's language, choices, attitudes and actions; (2) reflect upon their difficulties in managing a discussion and receive suggestions about how to face micro-situations of interaction; (3) express their own difficulties, doubts, awareness.

The collectively-written critical analysis is a particularly important methodological tool for the development of the teacher's awareness: divergent comments to a micro-situation lead to grasp a range of possible interpretations of both behaviours and interventions enacted; converging comments enable one to amplify the critical points of the management of the activity, on which it is necessary to (re)construct competences and refine one's sensitiveness.

We wish to underline the determinant conditions for the effectiveness of our MT approach. One first condition is the non-episodic nature of the situations for reflection and exchange: by progressively accumulating these moments of autonomous and interactive reflection, characteristic of our methodology, the teacher becomes more receptive and, in the long term, is led to develop new conceptions, attitudes and ways of acting. Another fundamental condition, crucial for the teacher's development process, is the enactment of a relationship between the members of a team, based on mutual trust, and the construction of a sense of belonging to a group that shares common values.

Moreover, the analysis of several MTs related to the implementation of a path designed with the teachers and aimed at the development of students' proving ability through algebraic language, allowed us to identify the *specific characters* which constitutes the profile of an 'effective teacher', who *poses him/herself as a model of aware and effective attitudes and behaviours for students* (Cusi & Malara 2009). The defining elements of this model, are as follows: the teacher must (a) be able to assume the role of "*investigating subject*", stimulating an attitude of research on the problem being studied, and of an *integral element* of the class group in the research being activated; (b) be able to assume the role of *operational/strategic leader*, through an *attitude towards sharing* (as opposed to transmission) of knowledge, and as a *thoughtfulness leader* in identifying efficient operational/strategic models during class activities; (c) be aware of his or her responsibility in maintaining a *harmonized balance between semantic and syntactic aspects* during the collective production of thought through algebraic language; (d) seek to *stimulate and provoke* the *building of key skills* in the production of thought through algebraic language (be able to generalize, translate, interpret, anticipate, manipulate), acting as an *"activator" of algebraic processes (generalization, traslation, manipulation, interpretation, anticipation)*; (e) also have the aim to stimulate and provoke *meta-level attitudes*, acting as an *"activator" of thoughtful attitudes* and *"activator" of meta-cognitive acts*, with particular reference to the control of the global sense of the processes.

The work developed with traeenes teachers (Cusi & Malara 2011), suggested us to conceive this construct as a possible theoretical lens for the analysis of classroom discussions to be used in secific workshops for/with in service teachers. In the future we wish to verify the effectveness of this construct also as a tool for the teachers' self analysis.

4. CONCLUDING REMARKS

In this paper we presented a brief overview of the literature and we sketched out some research results which offer meaningful indications about recent points of views on generalization processes. Then we focused our attention on some recent studies about generalization activities in early algebra teaching describing the position of some scholars.

In this frame we have considered the issue of the role played by the teacher in leading the students to engage in this kind of activities and through some short excerpts of classroom work we have shown the sharp relationship between the teacher's actions and the students' behaviours. We have also sketched a profile of a teacher who acts as an effective guide for the students to promote the development of a meaningful and aware approach to algebraic thinking.

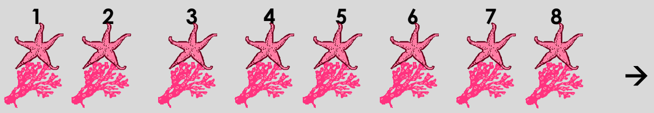
To conclude we stress the importance of the teacher's awareness at different levels to gain consciousness and control about the effective ways of posing him(her)self in the class and, above all, we underline the need of a refined teacher's education on this delicate aspect of teaching which requires a deep study of classroom episodes and above all a systematic careful self-analysis of the teacher's own practice.

APPENDIX

A problem situation presented in primary school (grade IV)

In the great reef life is very intense. You can possibly meet several types of animals: sponges, jellyfishes, octopuses, multicolour fishes. In the far eastern part of the reef a very numerous family of sea stars lives, each of them attached to a coral:

Alessia Angela Elena
Loretta Patrizia ...



When the new moon arises the sea stars shift and change the coral following a very old rule. Try to discover the rule looking at how the sea stars in the first positions move: Alessia goes to n° 3; Loretta goes to n° 5; Angela goes to n° 7; Patrizia goes to n° 9 Elena goes to n° 11

1) On the n. 78 coral the little star Valeria lives: which will be the number of the coral on which it will move? 2) Which will be the number of the coral where the sea star living at the 459th place will move? Justify your answers.

The discussion (the teacher's interventions are in italic)

At the beginning some pupils give numerical answers without any justifications or by chance

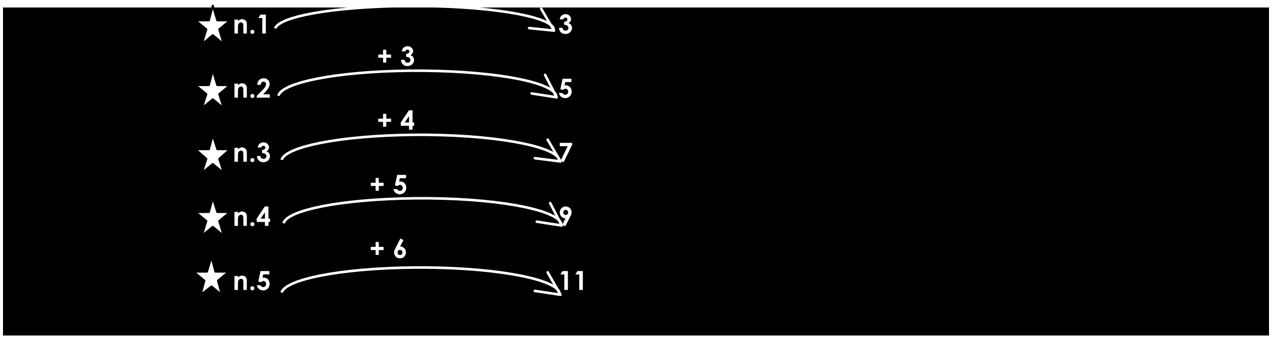
1. Teacher «*I asked you to justify your answers*»

Alex «the stars move: from 1 to 3, from 3 to 5, then 'plus 2', from 5 to 7 'plus 2'…»

Alessia: «I have added the number that says how much all the stars move: 2, 3, 4, 5… because from 1 to 3, it is +2; from 2 to 5, it is +3, then it moves from 3 to 7, it is +4; and then from 4 to 9, it is +5, from 5 to 11: +6 and adding 2+3+4+5 we obtain 15»

Beatrice «I have done in this way». (She goes to the blackboard and clearly describes her reasoning representing all the various cases with arrows) « The star Alessia has to move from place 1 to place 3 and then it is +2; Loretta has to move from n. 2 to n. 5, it makes +3; Angelica moves from n. 3 to n. 7, it makes +4; Patrizia moves from n. 4 to n. 9, it makes +5; Elena moves from n. 5 to n. 11, it makes +6. Then, in my opinion, [the answer for the coral n. 78] is 78+79, that is 157, because I have added to the number of the place of the star in the initial position, the number which follows it »

+ 2



2. Teacher *«Really good! What do you think about this? One of you said that Valeria arrives at n.80, another one said at n. 93, another one at 84, another one at 157»*

Nicola «I have not understood well Beatrice' s reasoning»

3.Teacher «*Beatrice, you have to help Nicola (*and addresses the class*), whether you do not understand, you ask* »

Beatrice «Yes. The star Alessia stayed at n.1 and she moved to n.3 … » (Beatrice starts from the first sea star and she retraces the arrow oriented from 1 to 3, she continues analogously with the other stars, indicating them while she is speaking).

4. Teacher *«What has Beatrice done with respect to the classmates who have spoken before her?»*

some pupils: «She has represented … ». Others: «She has outlined a scheme… »

(The teacher suggests Beatrice to write in red the value of the arrow operators. While Beatrice colours she explains)

Beatrice «… then for getting to 5 the star 2 makes +3; then from 3 for getting to 7, I have added 4; from 4 for getting to 9 I have added 5, from 5 for getting to 11 I have added 6»

Nicola «She has to put 6 because it is 5+1; she has to put the [number of] the star's address plus 1. She has to add "the address number plus 1" to "the address number" »

5. Teacher *«Good! Translate it into mathematical language »*

Nicola «+5+5+1»

Nicola, Beatrice and some others enrich the blackboard with a new representation: each arrow of the previous representation is splitted in two, the first arrow appears to be a variable operator depending on the place number and the second arrow appears to be the invariant operator '+1'.

6. *Teacher «Then if the star starts from 78, what will be its new place?*

Beatrice: «78+78+1=157»

7. Teacher: *(shaking hands with Beatrice. Then, addressing the class) «Have you understood?»*

Giulio «Then it has to go to number 157… I have written only the process: 78+78+1»

8. Teacher: « *Would it be possible to write the same thing in different ways?»*

Alex, Enrico, Nicola e Giulio give these writings.

78+78+1=157; 78+79=157; 78×2+1; 78+(78+1)

9. Teacher «*Very good. There is a new challenge for you: The star Filippa is at place n. 100; where does it move to?»*

Alessia «100×2+1=201»

10. Teacher «Ok. The star Maria is at n. 300; where does it move to?»

Alex «300+300 is 600, plus 1 that equals 601»

Beatrice «Or rather you can multiply its value times 2 and then plus 1 »

11. Teacher *«You have been very smart! We have not got to the generalization yet, but we are near*

Some days after the class restarts the activity.

12. Teacher «Go back to where we had stopped: which rule does the star Valeria follow to move to the new coral?»

Some pupils: 78+78+1=157. One of them rewrites this expression on the blackboard

13. Teacher «*Someone has said 78×2+1=157, do you remember? Now tell me: if a little star starts from n. 15 where is it when it arrives?»*

Pupils «15×2+1!»

14. Teacher «*Ok. And if it starts from 103?»*

chorus «103×2+1!»

The teacher picks other starting numbers: 598; 3654; 92045; she writes in column the pupils' sentences, *purposely leaving a space between the number of the starting coral and the chain of the operators acting on it:* 78 ×2+1; 15 ×2+1; 103 ×2+1; 598 ×2+1; 3674 ×2+1; 92045 ×2+1

Chorus «Times 2 plus 1, it remains the same!!!»

15. Teacher «*Excellent! '×2+1'remains constant. Now try to express in Italian the rule of this moving. We have to write the "Regulation of the sea stars movings". Imagine that the star Carlotta arrives at the colony for the first time. When there is the new moon it notices that all the sea stars move and change their place, she does not understand anything and she asksher neighbour star what she has to do. In your opinion which help can the neighbour star give her?*»

Alex «She has to do the number of its coral times 2 plus 1.»

16. Teacher «How can you say it in another way?»

Costanza « From the number of her house you have to go forward times 2 plus 1»

Piero «I shall say: if you are in the coral house number 50, you have to move to… you have to go… yet 50 house more and plus another»

17. Teacher «Meanwhile the little star starved … .Listen to me, we need to assign some names; how do we call these numbers? *(she indicates the first term of each sentence)*»

Costanza «Number of the house »

18.Teacher «Both of them are numbers of house »

Lucia «Number of the coral »

Chorus «Starting number»

19.Teacher «How do we call these in a competition?»

Chorus «Start! Arrival!»

(The teacher writes on the blackboard, respectively on the left-hand side and on the right-hand side of the sentences: "number of the starting coral"; "number of the arrival coral")

20. Teacher «*I suggest you to begin from the number that is after the equal sign*. (She says) "*The number of the arrival coral is equal …* »

Enrico «… to the starting number times 2 plus 1 »

Alessia «the number of the arrival coral is equal to twice the starting number plus 1»

21. Teacher «*We can take away "of the coral". Dictate it to me* »

pupils: «The arrival number is twice the starting number plus 1»

(The teacher writes the rule on the blackboard and reads it.)

22. Teacher «*Do you know how to translate it for Brioshi?*[21]»

Matteo «Times 2»

23. Teacher «*Only so? In your opinion Does Brioshi understand*?»

Mattia «78... »

24. Teacher «*Then does it hold true only for 78?»*

Enrico «It holds true for any starting number »

25. Teacher «*The idea is excellent, but in mathematics, after several studies, it has been decided to call 'any number' only with a letter* »

Mattia «I had said it!»

26 Teacher «*What do we choose as starting number*?»

Chorus «s»

27. Teacher «*And as arrival number*?»

Chorus «a»

Anna gives the rule in formal terms: «s×2+1=a» The class writes the relationship to be sent to Brioshi*:* s×2+1=a

Comment

At a first reading of this discussion, the teacher's behaviour can appear good. But in actual fact she does not act well. She speaks only with few pupils, she does not promote any interaction in the class, and above all she does not relaunch the validation of the pupils' proposals to the classmates. She does not take into account pupils' contributions which offer elements of discussion and of comparison (see Alex's proposal and Alessia's proposal). She expresses judgments through exclamations or emphatic gestures (intervention 2, intervention 7). She immediately directs the class towards the solution she had foreseen, as soon as it appears

[21] Brioshi is a virtual Japanese student who exchanges messages in mathematical language with pupils. His acknowledged skill in this area, encourages pupils to check the correctness of the mathematical expressions to be sent out to him.

(intervention 5-7)[22]. She disregards to enhance important contributions, even expressed in general terms, as the one by Nicola, which facilitates the emergence of the link between the initial and final coral-house of a sea star. Yet, she does not re-examine with the class the reasonings developed for sharing, pinpointing and consolidating them, but she limits herself to ask "did you understand?". She does not pose herself in a reflective way in front of the pupils, trying to help them overcome the procedural vision induced by the arrows representation, for instance discussing with the class about which coral-house they have to speak, the 'regulations' they have to write, so that the pupils can understand they have to write a verbal sentence related to the number of the final coral-house. She disregards the opportunity offered by Alex's intervention to clarify that a rule cannot be a simple procedure but it has to be a sentence with a complete meaning, forcing in this way a verbal representation of the sought rule. Trying to solve the question of the verbal representation of the relationship at stake, she poses a vague question (intervention 16) which does not allow pupils to face this delicate step, impossible to be done without a careful mediation of the teacher, where they have to shift from the number of the starting coral-house to the number of the final one. Yet, she does not bring the pupils to make explicit in the various numerical cases what the starting and final numbers represent, fact that prevents the pupils from formulating verbally a rule through the interpretation of the arithmetical sentences, rule in fact suggested by her (intervention 20). Moreover she does not face in a constructive way the question of how to introduce the letters as representation of the variables "number of the starting coral-house" , "number of the final coral-house", but only suggests their possible use. So, even if the algebraic representation of the rule is made in the class , this discussion does not allow the pupils to consciously understand the meaning of the algebraic expression.

Globally the discussion shows a bigger tension of the teacher for the attainment of her goal in a short time than for the care to appropriately address the pupils, educating their ways of seeing and facilitating the interaction among them; such a tension brings her to assume a procedural behaviour and to give scarce attention to the meanings associated with the actions in the various steps of a generalization process.

REFERENCES

Giovannini, A. (alias Enriques, F.) 1942, L'errore nelle matematiche, *Periodico di Matematiche*, serie IV, XXII, (pp. 57-65)

Blanton, M. and Kaput, J.J. (2011). Functional Thinking as a Route Into Algebra in the Elementary Grades. In J. Cai and E. Knuth (Eds.), *Early algebraization, A Global Dialogue from Multiple Perspectives* (pp.483-510). Advances in Mathematics Education: Springer.

Carpenter, T. and Franke, M. L. (2001). Developing algebraic reasoning in the elementary school: generalization and proof. In E. Chick et Al. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of algebra'* (vol. 1, pp. 155-162). Melbourne: University of Melbourne.

Carpenter, T.P., Franke, M.L. and Levi, L. (2003). *Thinking Mathematically. Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Carraher, D., Brizuela, B., and Schliemann, A. (2000). Bringing out the algebraic character of Arithmetic: instantiating variables in addition and subtraction, Nakahara,T, Yoyama, M (Eds.), *Proc. PME 24,* vol. 2, (145-152), Hiroshima: University of Hiroshima.

Carraher, D., Brizuela, B., and Darrell, E. (2001). The reification of additive differences in early algebra, in Cick, E., Stacey, K, Vincent, J., Vincent J. (Eds.), *The future of the teaching and learning of algebra*, (pp.163-170), Melbourne: University of Melbourne

Carraher, D., Martinez, M., and Schlieman, L. (2008). Early algebra and mathematical generalization, *ZDM*, 40, (pp. 3–22)

Cooper, T.J. and Warren, E. (2011). Years 2 to 6 students' ability to generalize: models, representations and theory for teaching and learning. In J. Cai and E. Knuth (Eds.), *Early*

[22] After the first intervention by Beatrice the teacher should have relaunched to the class the validation of the girl's reasoning, or at least she should have asked Beatrice to better explain why in her opinion 79 had to be added to 78, helping the class focus their attention on the extension of the regularity detected by the girl and trying to force her to express the relationship between the two numbers at stake (the number to be added to the number of the first coral-house is its successive), fact which allows to easily identify the relationship between the numbers of the two coral-houses of the sea stars.

algebraization, A Global Dialogue from Multiple Perspectives (pp. 483-510). Advances in Mathematics Education: Springer.
Cusi, A. and Malara, N.A. (2008). Approaching early algebra: Teachers' educational processes and classroom experiences, *Quadrante,* vol. XVI, n.1, (pp. 57-80)
Cusi, A. and Malara, N.A. (2009). the role of the teacher in developing proof activities by means of algebraic language, In M. Tzekaki et Al. (a cura di), *Proc. PME 33*, vol. 2, (pp. 361-368), Thessaloniki.
Cusi, A. and Malara, N.A. (2011). analysis of the teacher's role in an approach to algebra as a tool for thinking: problems pointed out during laboratorial activities with perspective teachers, in Pytlak, M., Swoboda, E. (Eds.) *CERME 7 proceedings*, (pp. 2619-2629), Rzezsow: University of Rzezsow.
Cusi, A. and Malara, N.A. (to appear). Educational processes in early algebra to promote a linguistic approach to it: behaviours and awareness emerged in teachers, *Recherches en Didactique des Mathématiques*
Cusi, A., Malara, N.A., and Navarra G. (2011). Early Algebra: Theoretical Issues and Educational Strategies for Promoting a Linguistic and Metacognitive Approach to the Teaching and Learning of Mathematics. In J. Cai and E. Knuth (Eds.), *Early algebraization, A Global Dialogue from Multiple Perspectives* (pp. 483-510). Advances in Mathematics Education: Springer.
Cusi, A. and Navarra, G. (2012). Aspects of generalization in early algebra, *in these proceedings*
Dorfler, W. (1989). Protocols of actions as a cognitive tool for knowledge construction, in Artigue, M., Rogalski, J. Vergnaud, G. (Eds), *proc. PME 13*, vol. 1, (pp. 212-219), Paris.
Dorfler, W. (1991). Forms and means of generalization in mathematics, in A. Bishop et alii (Eds.), *Mathematical Knowledge: Its growth through teaching*, Kluver , 63-95
Dorfler, W. (2008). En route from patterns to algebra: comments and reflections, ***ZDM,*** 40 (1), 143-160
Ellis, A. (2007). A Taxonomy for categorizing generalizations: generalizing actions and reflection generalizations, *Journal of Learning Science*, 16:2, (pp. 221-262)
Ferrari, P.L.: 2006, 'From verbal texts to symbolic expressions: A semiotic approach to early algebra', in Novotná, J., H.Moraová, M.Krátká, N.Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Prague, vol.3, 73-80.
Fuji, T. and Stephens, M. (2001). Fostering understanding of algebraic generalization through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, Jl. Vincent and Jn. Vincent (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of Algebra*', vol. 1 (pp. 258-264). Melbourne.
Hejny, M. (2003). Understanding and structure, Mariotti, M. (Ed.), *proc. CERME 3*, Bellaria, WG3-(pp.1-8)
Jaworski, B. (1998). Mathematics teacher research: process, practice and the development of teaching. *Journal of Mathematics Teacher Education*, 1, 3-31.
Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54, 249-282.
Jaworski, B. (2006). Theory and practice in mathematics teaching development: critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187–211
Kaput. J. (1995) A Research base supporting long term algebra reform? Proc. PME 17-NA Chapter
Kaput. J. (1999). Teaching and learning a new algebra with understanding, in E. Fennema & T. Romberg (Eds.), *Mathematics classroom that promote understanding*, Mahwah, NJ: Lawrence Erlbaum Associated, Inc.
Kaput J., and Blanton M.: 2001, Algebrafying the elementary mathematics experience: transforming task structures, in Chick, H., Stacey, K., Vincent Jl. and Vincent, Jn. (Eds.), *Proc.*

12th ICMI Study 'The future of the teaching and learning of Algebra', vol. 1, (pp. 344-353), Melbourne: University of Melbourne

Kaput, J., Carraher, D. and Blanton, M. (Eds.) (2007). *Algebra in the early grades* (pp. 95-132). New York: Erlbaum.

Lee. L. (1996). An initiation into algebraic culture through generalization activities, in Bednardz, N. Kieran, C., Lee, L. (Eds.), *Approaches to algebra*, (pp. 87-106), Dordrech: Kluver

Lerman, S. (2001). A review of research perspectives on mathematics teacher education. In T. J. Cooney e F. L. Lin (Eds.), *Making sense of mathematics teacher education* (pp.33-52). Dordrecht: Kluwer.

Malara, N.A. (2003). Dialectics between theory and practice: theoretical issues and aspects of practice from an early algebra project. In N.A. Pateman , B. J. Dougherty e J. T. Zilliox (Eds.), *Proceedings of PME 27*, vol.1 (pp.33-48). Honululu, USA.

Malara, N.A. (2005) Leading In-Service Teachers to Approach Early Algebra, In Santos, L. (Ed.) *"Mathematics Education: Paths and Crossroads"*, (pp. 285-304), Lisbona: Etigrafe

Malara, N.A. (2008). Methods and Tools to Promote in Teachers a Socio-constructive Approach To Mathematics Teaching. In Czarnocha, B. (Ed.), *Handbook of Mathematics Teaching Research* (pp. 273-286). Rzeszów University Press.

Malara, N.A. and Navarra, G. (2001). "Brioshi" and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language. In Chick, H., Stacey, K., Vincent Jl. and Vincent, Jn. (Eds.), *Proceedings of the 12th ICMI Study 'The future of the teaching and learning of Algebra*', vol. 2 (pp.412-419). Melbourne: university of Melbourne

Malara, N.A. and Navarra, G. (2003). *ArAl Project: Arithmetic Pathways Towards Favouring Pre-Algebraic Thinking*. Bologna: Pitagora.

Malara, N.A. and Navarra G. (2005). Approaching the distributive law with young pupils, in *proc. CERME 4* (Saint Feliu de Guixol) or rivista *PNA Revista de Investigacíon en Didáctica de la Matematica*, 2009*,* vol. 3, n. 2, (pp. 73-85)

Malara, N.A. and Navarra G. (2011). Multicommented transcripts methodology as an educational tool for teachers involved in early algebra, in Pytlak, M., Swoboda, E. (eds.) *CERME 7 proceedings*, Università di Rzezsow, Polonia , (pp. 2737-45)

Mason, J. (1996a), Future for Arithmetic & Algebra: Exploiting Awreness of Generality, in Gimenez, J., Lins, R., Gomez, B. (eds), *Arithmetics and Algebra Education, Searching for the future*, (pp. 16-33) Barcelona: Universitat Rovira y Virgili.

Mason, J. (1996b). Espressing generality and roots of algebra, in Bernardz, N., Kieran, K, Lee, L. (eds), *Approaches to Algebra*, (pp. 65-86) Dordrecht: Kluver Academic Publisher.

Mason, J. (1998). Enabling Teachers to Be Real Teachers: Necessary Levels of Awareness and Structure of Attention. *Journal of Mathematics Teacher Education*, 1, (pp. 243-267).

Mason, J. (2002). *Researching Your Own Practice: the Discipline of Noticing*, London: The Falmer Press.

Mason, J. (2008). Being Mathematical with and in front of learners. In Jaworski, B. Wood, T. (Eds.), *The Mathematics Teacher Educator as a Developing Professional*, (pp. 31-55). Sense Publishers.

Mason, J., Grahm, D, Pimm, D. and Gower, N. (1985). *Route to/roots of algebra*, Open University, Milton Keynes

Orton A. and Orton J. (1994). Students' perception and use of pattern and generalization. In Da Ponte, J., Matos, J.F. (eds) *proc. PME 18*, vol.3. (pp. 407-414) Lisbon: University of Lisbon

Orton, J. and Orton, A. (2006), Making Sense of Children's patterning, in Puig, L., Gutierrez, A. (eds), proc. PME 20, vol.4 (pp. 83-90), Valencia: University of Valencia

Orton, A. and Orton, J. (1999). Pattern and the approach to algebra, in Orton A. (ed), *Pattern in the Teaching and Learning of Mathematics*, (pp. 104-120), London: Contunuum

Radford, L. (1996). Some reflection on teaching algebra through generalization, in Bernardz, N., Kieran, K, Lee, L. (eds), *Approaches to Algebra*, (pp. 107-111) Dordrecht: Kluver Academic Publisher.

Radford, L. (2001). factual, contextual, and symbolic generalizations in algebra, in in van der Huenvel-Panhuizen. M. (ed), *Proc. PME 25*, (vol.4., pp.81-88), Utrecht: Freudenthal Institute

Radford, L. (2003). gestures, speech, and the spouting of signs: a semiotic culturale approach to students' type of generalization, mathematical thinking and learning, *Mathematical Thinking and Learning*, 5 (1), 37-70

Radford, L. (2006).Algebraic thinking and the generalization of paterns: a semiotic perspective. In Alatorre, S., Cirtina, J., Sáiz, M., Méndez, A. (eds) *Proc. PME 28-NA* Chapter (vol.1 pp. 2-21). Mexico:UPN

Radford, L. (2008). Iconiciy and contraction: a semiotic investigation of forms of algebraic generalizations of patterns in different context. *ZDM*, 40 (1), 83-96

Radford, L. (2009). Signs, gestures, meanings: algebraic thinking from a cultural semiotic perspective, Durand-Guerrier, V. , Soury-Lavergne, S. Arzarello, F. (eds) , *proc. CERME 6*, Lyon, XXXIII – LIII:

Radford, L. (2010), The eye as a theoretician: seeing structures in generalizing activities, *For the Learning of Mathematics* **30**, 2 (pp. 2-7).

Radford, (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai and E. Knuth (Eds.), *Early algebraization, A Global Dialogue from Multiple Perspectives* (pp.483-510). Advances in Mathematics Education: Springer.

Rivera, F. (2010) Visual Templates in pattern generalization activity. *Educational Studies in Mathematics*, 73(3) (pp. 297-328)

Rivera, F. and Rossi Becker, J. (2007), Abductive, inductive (generalization) strategies of preservice elementary majors on patterns in algebra, *Journal of Mathematical Behaviour*, 26(2) (pp.140-155)

Rivera, F. and Becker, J. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. ZDM, 40(1), (pp. 65–82).

Rivera, F. and Rossi Becker, J. (2011), Formation of pattern generalization involving linear figural patterns among middle school students: results of a three-year study. In J. Cai and E. Knuth (Eds.), *Early algebraization, A Global Dialogue from Multiple Perspectives* (pp.323-366). Advances in Mathematics Education: Springer.

Sfard, A. (1991): On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Math*. 22, (pp. 1 – 36)

Schliemann, A,D., Carraher, D.W. and Brizuela, B.M. (2001), When tables become function tables, in van der Huenvel-Panhuizen. M. (Ed.), *Proc. PME 25*, vol.4., (pp.145-152), Utrecht

Sfard, A. (2005). What could be more practical than good research? On mutual relations between research and practice of mathematics education. *Educational Studies in Mathematics*, 58(3), (pp. 393 – 413)

Shoenfeld, A. (1998). Toward a theory of teaching in context. *Issues in Education*, 4(1), (pp. 1-94).

Stacey, K. (1989). Finding and using patterns in linear generalizing problems, *Educational Studies in Mathematics*, 20, (pp. 147-164)

Warren, E. (2006). Teacher actions that assist young students write generalization in words and in symbols. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proc. PME 30*, Vol. 5, (pp. 377-384). Prague.