EDUCATIONAL PROCESSES IN EARLY ALGEBRA TO PROMOTE A LINGUISTIC APPROACH: BEHAVIOR AND EMERGING AWARENESS IN TEACHERS

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PROCESSUS DANS L'ENSEIGNEMENT PRE-ALGEBRIQUE POUR PROMOUVOIR UNE APPROCHE LINGUISTIQUE : COMPORTEMENTS ET VIGILENCE EMERGEANT CHEZ DES ENSEIGNANTS

Résumé – Nous faisons un état des changements de perspectives survenues dans l'enseignement de l'algèbre. En suivant les tendances actuelles de l'enseignement qui considèrent le rôle complexe de l'enseignant, nous présentons le projet (ArAl) comme un corpus d'études pour l'approche du pré-algèbre (early algebra) de façon linguistique et socio-constructiviste et comme un système intégré de formation des enseignants. Nous développons la méthodologie adoptée dans le projet avec notamment les transcriptions multi-commentées des discussions en classe (MT) comme des outils propres à affiner la capacité des enseignants à avoir une observation critique de leur action en classe sur des séances privilégiant l'approche au pré-algèbre.

Mots clés: pré-algèbre, enseignement socio-constructif, formation des enseignants, réflexion critique des enseignants, projet d'ArAl, transcriptions de classe multi-commentée.

PROCESOS EDUCATIVOS EN ÁLGEBRA PRECOZ PARA PROMOVER UN ACERCAMIENTO LINGÜÍSTICO: COMPORTAMIENTOS, Y CONOCIMIENTOS SURGIDOS DE LOS ENSEÑANTES

Resumen – Se revisarán los cambios de perspectiva que se han producido en la enseñanza del álgebra, deteniéndonos en las tendencias actuales de la enseñanza y se considerará el papel complejo del enseñante. Se introducirá el proyecto (ArAl) cómo corpus de estudio para un acercamiento al álgebra precoz de tipo lingüístico y socio-constructivo y como sistema integrado de formación de los enseñantes. Nos centraremos en la metodología adoptada en el proyecto e introduciremos las transcripciones con intercambio de comentarios de las discusiones en clase (MT) como instrumentos idóneos para afinar la capacidad de los profesores a la hora de dirigir debates de acercamiento al álgebra precoz a través de la observación critica de su propia actuación en clase. Se utilizará un extracto del MT y se analizarán los

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comentarios al respecto resaltando los comportamientos y los conocimientos que han surgido en el enseñante. Concluiremos con un breve comentario sobre el valor de esta metodología y las condiciones que determinan su eficacia.

Palabras-claves: álgebra precoz, enseñanza socio-constructiva, formación de enseñantes, reflexiones críticas de los enseñantes, proyecto ArAl, transcripciones de las clases con intercambio de comentarios.

ABSTRACT

In this paper, we trace the changes in perspective that occurred in algebra teaching, sketching recent trends and discussing the increasingly complex role played by the teacher. We introduce our project (ArAl) both as a corpus of studies for a linguistic and socio-constructivist approach to early algebra and as an integrated system of teacher education. We present the methodology adopted in the project and introduce the multi-commented transcripts (MT) of class discussions, seen as a useful tool to promote teachers' ability to orchestrate early algebra discussions and to critically observe their own actions in the classroom. We provide an MT excerpt and analyze the related comments, highlighting behavior and awareness emerging in the teacher. We conclude with some remarks about the value of our methodology and the conditions that determine its efficacy.

Keywords: early algebra, socio-constructivist teaching, teacher education, teachers' critical reflection, the ArAl project, transcripts of multi-commented classes.

WHY EARLY ALGEBRA

With the advent of the new math, educational research began to focus on a richer and more multifaceted vision of the teaching of algebra, moving away from purely formal aspects and aligning more closely with the constitutive processes of the discipline itself. Key elements of this vision are both the identification and the study of relationships emerging from the observation of real phenomena. The vision led to an appreciation of the use of algebraic language for representing and deducing information.

In this regard, several outstanding research studies in England in the 1980s proposed pathways for innovation based on successful teaching experiments. Those pathways generally concerned different types of algebraic activities in a wide range of contexts that supported the student's access to generalization, modeling, justification, and proof (Bell, Malone, &Taylor 1987) and that required the activation of the so-called essential algebraic cycle: "represent, manipulate, interpret" (Bell 1996).

In the same period, Chevallard (1985, 1989, 1990) faced the problem of the transition from arithmetic to algebra. In a wide and refined study, he analyzed historical-epistemological aspects and reflected on the key moments of that transition. He then presented the innovations introduced by modern mathematics, referring both to the numerical sets point of view and to the functional one. In this frame, he conceived of *modeling* as a unifying notion for mathematical activities at all school levels, also applied to science or the real world. Moreover, he considered natural numbers as a privileged "object of study," stressing its value as a special domain where students can conjecture arithmetical properties, model, and prove them through syntactical transformations.[1]

Later, Kieran (1996) widened the usual vision of school algebra and inserted the classic transformational activities between two other

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[1] In particular, Chevallard (1990) underlined how "l'emploi fécond du calcul algébrique *suppose une dialectique permanente entre le calcul et les phénomènes qu'il permet de modéliser*, entre la *conduite* du calcul et le *raisonnement* qui permet seul d'arriver aux décisions de calcul pertinentes" (p. 22). But he considered the didactical integration of algebraic modeling highly problematic because of the « curriculum caché (hidden curriculum), ensemble des conditionnes et contraintes qui entrent dans la définition du curriculum, mais qui ne sont pas explicitées par le textes qui administrent officiellement le système d'enseignement » (p. 36).

types of activities: generational activities (which concern representation and interpretation of situations, properties, and relations) and global meta-level activities (for which algebra is used as a tool but that are not exclusive to algebra).

In those years, the debate was mainly about the relationship between arithmetic and algebra in teaching. Some argued for the importance of looking at the two domains from a perspective of continuity, stressing their mutual synergies (Sadovsky 1999); others underlined the epistemological rupture between the two areas and the unavoidable didactical cuts (Filloy & Rojano 1989) and consequent cognitive jumps (Herscovics & Linchevski 1994). However, there was general agreement on the need to set the foundations of the teaching of arithmetic in a relational and pre-algebraic perspective. The objectives were as follows: (a) to overcome classical difficulties and stereotypes, such as the directional equal sign or the lack of closure in arithmetic expressions, in order to favor both cognition and control of the plurality and variety of arithmetical representations of one number; (b) to induce a structural vision of arithmetic expressions, detecting analogies and differences in a flexible way and being aware of the role played by the properties of the operations; and (c) to open the way to generalization processes; to the solution of problems and equations through naïve exploratory procedures, aiming—through reflection on the enacted strategies—at the construction of appropriate cognitive schemes to be used to understand the genesis of the objects of algebra as well as to appreciate their value.

The acknowledgement of the importance of these issues led to envisaging a specific area of teaching, pre-algebra, as a sort of buffer between elementary number theory and algebra (Linchevski 1995).

In those years, different scholars underlined how important it was that students acquire *symbol sense* (Arcavi 1994) through manifold activities that enabled them to reach abilities, understanding, and ways of feeling that might lead them to act in a flexible and instinctive way within a system of symbols, to move through wider or different systems of symbols and coordinate interpretations of formulas in various solution contexts (Arzarello, Bazzini, & Chiappini 1993, Filloy 1991, Gray & Tall 1993, Kaput 1991, Lins 1990). Later, Radford (2000) proposed a description of the learning of algebra as an appropriation of a new way of acting and thinking interwoven with the novel use and production of signs, which he interpreted "as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage," shifting the focus "from what signs represent to what they enable us to do."

Many studies were carried out everywhere in the world within that framework, with the aim to test the validity of the hypotheses and the possibility of actually implementing them in the classroom. Teaching experiments were initially carried out with 11–13-year-old pupils on generational algebraic activities and later moved to primary school, especially in the United States (Blanton & Kaput 2001, Carraher 2001, Carpenter, Franke, & Levi 2003). Among those studies, some stand out for the theorization of models of conceptual development of a socio-constructive type in algebra: They emphasize the impact of the classroom environment, as well as the important role played by the teacher, in the framework of a vision of algebra as a language. These studies mainly concern the implementation of innovative activities in primary school and beginning of secondary school, analyzing both the behavior and the learning of pupils engaged in pre-algebraic and algebraic activities, but they also deal with the issue of a suitable teacher education by documenting interventions aimed at teachers' professional development (for further discussion, see Cusi, Malara & Navarra (2011) and the related bibliography).

Our research studies are located within the latter trend and develop within the *ArAl Project:*[2] *Paths in Elementary Number Theory to Favour Pre-Algebraic Thinking* (Malara & Navarra 2003) that we discuss in the following section.

All these studies form the constituting corpus of *early algebra,* a disciplinary area nowadays acknowledged throughout the world, as the recent books *Algebra in the Early Grades* (Kaput, Carraher & Blanton 2007) and *Early Algebraization* (Cai & Knuth 2011) demonstrate.

THE ARAL PROJECT

Starting in the early 1990s, we implemented some projects for the innovation in the teaching of algebra in junior high school, with the collaboration of small groups of teachers. Our results, in line with what researchers documented in other countries, highlighted that the roots of many difficulties lie in how pupils have been taught arithmetic.

Hence, aiming at constructing a pre-algebraic mental attitude in primary school pupils, we developed our *ArAl Project,* which proposes an approach to early algebra of a linguistic-constructive type and from a relational perspective starting in primary school.

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[2] *ArAl* is an acronym for « Arithmetic and Algebra ».

This project is based on the hypothesis that algebraic language and natural language might be learned with analogous learning modalities and that the needed mental schemes should be constructed from the very beginning, leading pupils to face increasingly finer pre-algebraic experiences. Therefore, generative pre-algebraic activities should be anticipated at the beginning of primary school, and even before that, in kindergarten, to favor the genesis of algebraic language, viewed as a generalizing language. Natural language plays a central role in our approach: (a) as the main didactical mediator for the slow construction of syntactic and semantic aspects of algebraic language; (b) during the processes aimed at a critical review of ideas (verbalization, argumentation, discussion, exchange); and (c) in setting up the bases for both producing and interpreting representations written in algebraic language (i.e., natural language becomes a tool to identify and make explicit algebraic thinking that is often "hidden" in concepts and representations in arithmetic). The activities with students are carried out to shift the focus from syntactical aspects to aspects related to the production of thought through a continuous reflection on the meaning of both the introduced signs and the enacted processes, in a way similar to that characterizing the learning of natural language, in order to help students progressively develop algebraic thinking in a strict intertwining with arithmetic.

Another central element in our approach to early algebra is the idea that the semantic bases for a genuine understanding of algebraic expressions could be built up by helping students recognize and interpret *canonical and noncanonical representations of numbers*.[3] The introduction of these constructs is aimed at helping students overcome those destructive beliefs that prevent them from conceiving of some expressions (e.g., 7 × 5) as possible representations of a number (35, in this example). Siety (2001 pp. 18–19) highlights this problem, stressing that it represents an origin of possible blocks in the learning of algebraic language and, in particular, in a meaningful understanding of syntactical aspects of algebra. In the ArAl project, we foster the conceptualization of a wide variety of possible representations of a number in order to help students develop an awareness that the choice of one representation over another should be related to its effectiveness in relation to the objective of the activity that is being carried out. The objectification of the constructs of

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[3] Among the possible representations of a number, one (e.g., 12) is its name, called its *canonical* form; all the others (e.g., 3 × 4, (2 + 2) × 3, 36/3, 10 + 2, …) are its *noncanonical* forms, and each of them will make sense in relation to the context and the underlying process.

canonical and noncanonical representations is also aimed at favoring the development of a relational vision of numbers and the conceptualization of the role of arithmetical operations in highlighting the equivalence of different representations of the same number through processes of syntactical transformations.

All this requires the definition of a new classroom culture through the construction of an environment that may informally stimulate the autonomous elaboration of what we call *algebraic babbling;* that is, the experimental appropriation of a new language whose rules are gradually located at their right place in a didactical contract that tolerates initial moments of syntactic confusion and that is open to reflection and to a comparison of the enacted representations.

In this approach, *the teacher has a key role*. In fact, she or he needs to set up a teaching strategy that allows for the implementation of an authentic, socially shared mathematical activity in which space is given to linguistic aspects, to the representation of information and processes, as well as to meta-cognitive aspects. The last are important for monitoring the appropriateness and suitability of representations, to recognize and identify equivalent ones and select the best. All this requires a deep restructuring of teachers' conceptions about both the content to be taught and the teaching methodology in the classroom: A real "culture of change" is entailed.

This is the reason that the project pursues a double objective: to promote innovative activities in the classroom and contribute to specific teacher education, by working and reflecting with them. Tools, methods, and activities set up within the ArAl project work as a support for teachers in proposing early algebra activities in the classroom, with the modalities described above, and in forming them professionally through reflection on their own action in the classroom (Cusi & Malara 2008, Cusi et al. 2011, Malara & Navarra 2009 and 2011). The ArAl Project has received both national and European awards because of the role it plays in scientific education. Many networks of schools in Italy take part in the project, planning their mathematics curriculum in reference to our activities.

In order to justify our choices related to teacher education, we briefly sketch our theoretical framework in the next section.

THE THEORETICAL FRAMEWORK FOR TEACHER EDUCATION

Beginning in the 1990s, research has focused on the teacher, underlining the manifold and delicate roles he or she must play, and pointing out some fundamental elements for the construction of an appropriate professionalism. The value of critical reflection by the teacher in order to acquire and strengthen those abilities is nowadays widely recognized.

Since the early 1980s, Schön (1983, 1987) theorized about the importance of teachers' reflection at different levels: on action, for action, in action. Later, many other scholars supported the value of teachers' critical reflection on their practice; they also stressed the value of practices based on sharing the outcomes of that reflection among teachers or, even better, among teachers and researchers (for bibliographic references and further discussion of these issues, see Malara & Zan 2008). In particular, Mason (1998, 2002) maintained that the education of a "real" teacher calls for the development of different levels of awareness: about action, about the discipline, about the educational project. For educating teachers to this awareness, he suggested the practice of the discipline of noticing, aimed at acquiring the skill of constantly and carefully monitoring oneself, recommending that one's notes should be shared with colleagues to be validated. Drawing on Schön's work, Jaworski (1998) claimed the importance of *reflective practice,* whose essence is described as making explicit teaching approaches and processes so that they can become objects of a detailed critical examination. She also objectified reflection itself as "socially and politically oriented action," whose product is *practice,* viewed as "committed action, deriving from education" (Jaworski 2003). She maintained that "communities of enquiry in teaching"—that is, mixed groups including teachers and researchers—are effective and emphasized that participating in a group is a "process of becoming" and leads the teacher to acquire an identity (Jaworski 2004). The international community of researchers has acknowledged that these activities of joint reflection foster teachers' professional development in relation to their capacity of having control over, and predicting the effects of, their own actions in the class (Davis, Brown, et al. 2009).

We agree with these standpoints, which, by the way, go along with the Italian model of research for innovation. Our hypothesis is that observation and a critical-reflective study of socio-constructive processes in the classroom is a necessary condition for the teacher to become aware of the new role he or she must play in the classroom, of the processes that develop in the collective mathematical construction, as well as of the variables involved. Through this kind of work, one could develop an increasing awareness in learning to transform thousands of occasionally noticed things into a tool of one's individual methodology. We also believe that this activity should be supported by the study of the results of mathematics education theory (that may strengthen the teacher's pedagogical content knowledge) and that the reflections produced should be shared among groups of teachers and researchers.

OUR METHODOLOGY OF WORK WITH AND FOR TEACHERS

The activities we carry out with the teachers involved in our project, conceived from a perspective of lifelong learning, are devoted to helping both novice and experienced teachers foster the sharing of their experiences. Those who choose to participate in ArAl teaching experiments are mainly motivated by their "first encounter" with the project through publications, congresses, or events in the schools. Often, these teachers have already studied the project and, in particular, its units[4] and the glossary that can be found on the project's website.[5] When they actually face the teaching experiment, the teachers nevertheless show uncertainty about class discussions, viewing them as open and unpredictable situations that are difficult to manage.

As a support to teachers and an answer to their needs, a mentor-researcher is associated with each group of teachers involved in the same teaching experiment: Teachers and mentor share some moments of work firsthand, together with a dialogical relationship via e-mail. There are also regular working sessions of small groups with their mentor-researcher, and also collective sessions involving all the researchers and teacher experimenters, all held in schools or at the university.

Believing that observation and a critical-reflective study of classroom-based processes help teachers become aware of the

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[4] The units can be viewed as models of sequences of didactical projects, open to the teacher's choices and focused on a specific strand of activities. They provide information on the mathematical meaning and the objectives of the single activities presented, reporting excerpts that exemplify class discussions, as well as comments on pupils' behavior (meaningful constructions, frequent attitudes, difficulties) and on teachers' behavior (appropriate interventions, ways of introducing and managing issues, attitudes, etc.).

[5] The units are supported by the theoretical framework and, most of all, by the glossary, available at the project's website <www.aralweb.unimore.it>, where teachers can find clarification and further material on mathematical, linguistic, psychological, socio-pedagogical, and methodological-didactical issues and also find prototype didactical sequences aimed at giving them a stimulus for their own elaboration of the highlighted teaching processes.

processes involved in every discussion and of the variables that determine those processes, our objective is to lead the involved teachers to: (a) become increasingly able to interpret the complexity of class processes through the analysis of the inner *micro-situations*, to reflect on the effectiveness of their own role and become aware of the effects of their own *micro-decisions*; (b) be in better and finer control of their behavior and the communication styles they use; and (c) notice, during classroom activity, the impact of the critical-reflective study undertaken on pupils' behavior and learning.

To achieve this objective, we involve the teachers in a complex activity of critical analysis of the transcripts concerning class processes and of reflection upon them, aimed at highlighting the interrelations between knowledge constructed by the students and the behavior of the teacher in guiding the students in those constructions. The analysis is carried out by building up what we call *multi-commented transcripts* (MT), or "the diaries." They are realized after transcribing in a digitally formatted text the audio recordings[6] of lessons on topics that were previously agreed on with the researchers. The MTs are completed by the teacher experimenters, who send them, together with their own comments and reflections, to the mentor-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers. Often, the authors make further interventions in this cycle, commenting on comments or inserting new ones. This methodology (MTM) is characterized by a sort of choral web participation because of the intensive exchanges via e-mail, which contribute to the construction of the MTs, and to the fruitfulness of the reflections emerging from the different comments.

The MTs become important working tools since they provide an overall picture of the teacher's action in the classroom and offer the chance to test the consistency between teaching practice and reference to the theory at stake (in both mathematics and mathematics education). They offer both teachers and researchers elements to empower the effectiveness of their interventions in the respective fields, and make it possible to perceive teachers' culture and attitudes.

[6] We chose to analyse audio recordings instead of videos of classroom processes because we believe (Malara & Zan 2008) that whereas watching the video might not enable teachers to completely capture the details of the verbal interaction, analysing transcripts instead fosters the crystallization of interactive processes and highlights gaps, crucial decision-making moments, and also omissions, oversights, and carelessness.

The comments also allow the teacher to develop skills and sensitivity and hence to improve the overall quality of his or her teaching action.

We next present an example of MT that illustrates the type of activity introduced in the classes, the teacher's way of working, the awareness that emerged after the analysis of the discussion, and her achievements during the activity.

AN EXAMPLE OF MT

The MT excerpt presented here concerns a class intervention located within a sequence of activities carried out in a Grade 6 class. The task on which the discussion is focused is meant to be inserted in an ArAl unit, still under investigation, on the multiplicative structure of natural numbers, on the relation of divisibility, and on relational aspects of numbers in general. The activities of the unit will partially reflect a path, which is synthesized in Malara (1999) and further discussed, focusing on the teacher's role, in Malara and Iaderosa (1999). This path develops, throughout the three years of lower secondary school, from activities similar to those that we shortly analyze below into more complex ones, such as the construction of proofs (see for example Malara & Gherpelli 1997 or Malara 1999).

As we stressed before, in our project we propose a relational approach to arithmetic, with a focus on additive, multiplicative and mixed representations of numbers, aiming at introducing students to the justification of arithmetical properties, also related to divisibility. Usually these contents are only partially proposed at school, where traditionally the factorization of natural numbers is introduced at grade 6 only from an operational point of view and is used later mainly in the sum of fractions. Therefore students do not have the possibility to appreciate both the role played by multiplicative representations and the simplification introduced in these kinds of expressions, thanks to the exponential notation. In this way, they do not have the opportunity to grasp the meaning of prime factors and to better highlight numerical properties.

Although our approach is in tune with the Italian programs for lower secondary school (starting from the innovative ones of 1979), these programs are often disregarded by teachers because of the negative influence played by the more widespread textbooks, mainly focused on mnemonic or computational aspects. For example, while many textbooks propose the classical properties of arithmetical operations, mainly as abstract laws, in our approach we work on the generalization of problem-solving processes aiming at fostering the conceptualization of different properties bringing students to conceive them as real theorems-in-action (see for example Malara & Navarra 2005).

As Booker (1987) also stresses, we believe that the predominance of multiplicative representations in algebra (see, e.g., the case of algebraic fractions or polynomials) and the difficulties faced by students in operating with multiplicative and exponential representations (Malara & Iaderosa 2000) make it necessary to focus on this kind of representation beginning in the first years of school. We are working on identifying a sequence, mainly constituted by modeling activities in mathematics, aimed at: (1) reformulating, in tune with the ArAl framework, typical arithmetical activities usually proposed for Grade 6 in order to encourage the reinterpretation of traditional tasks by those teachers who are reluctant to propose innovative activities in their classes; and (2) leading students to face modeling activities in arithmetic in order to make them aware that, thanks to the use of letters, it is possible to express relations and numerical properties in general terms.

The activities concern a naive introduction of letters in simple numerical problems involving one or more arithmetical operations and aimed at: (a) valuing the distinction between number and its representation; (b) identifying different additive and multiplicative representations of the same number; and (c) comparing numbers through their multiplicative representations. Particular attention is given to the coordination of additive and multiplicative representations such as: $a + a + a$, $3 \times a$, $3 \bullet a$, $a \times 3$, $a \bullet 3$ and $3a$; to the links between direct and inverse operations; and to the different ways of representing a given relationship between two numbers. In this sense, it is important to stress that, as documented in (Malara & Iaderosa 2000), in the ArAl project, instead of working separately on the additive and multiplicative structures of natural numbers, we work with students presenting the analogies between these two structures.

Through the proposed situations, pupils are guided to compare and transform given numerical relationships, to identify formal analogies, equivalences and differences between representations, to distinguish an unknown from a generic number, and to approach the concept of variable.

The problem discussed in the following excerpt is focused on the comparison between two given numerical expressions, each associated with a specific name: $\mathrm{R} = a \times 4 \times 9$; $\mathrm{S} = 2^2 \times b \times 3^2$. It is important to stress that R and S are introduced, in tune with our approach, as names given to the two expressions and that the symbol of equality used to introduce these two expressions has a definitive nature. In fact, $\mathrm{R} = a \times 4 \times 9$ and $\mathrm{S} = 2^2 \times b \times 3^2$ are not to be conceived of as formulas such as, for example, the one for the area of a rectangle in which the equal sign expresses the relationships between the three quantities involved.

The students are required to compare these two multiplicative expressions, each with a different literal factor, with reference to the value they get depending on the numerical values assigned to the letters. Being able to address the question posed ("What can we say about the two expressions R and S?") requires: (a) a good command of the exponential notation; (b) the ability to recognize the equivalence of expressions independently of the position of the terms they contain (referring to the properties of multiplication); and (c) the ability to compare expressions referring to the possible values of the terms they contain. The activity is aimed, in particular, at: (a) approaching a view of letters as variables, helping students gradually overcome the concrete stage of mere numerical substitution, shifting the focus from the single number to its numerical domain; (b) strengthening the idea that different letters may also have equal numerical values and therefore represent the same number; and (c) helping students focus on the communicative effectiveness of "giving names" to expressions.

We consider this problem open ended because it is conceived of as a "discussion problem," and different possible answers could be expected from students according to their different level of development in the use of letters: (a) R and S are different because *a* is different from *b*; (b) R and S are different because, if we substitute, for example, the value 5 for *a* and the value 7 for *b,* the result is different; (c) R and S are the same number if *a* is equal to *b*; and (d) R is a multiple of S if *a* is a multiple of *b* (this kind of answer is related to an exploration of the possible relationships between R and S). This kind of problem is typical of our approach to early-algebra. On the one hand, it both stimulates an aware approach to the multiplicative-exponential representation of natural numbers (helping students recognize equivalent representations without executing computations) and fosters a better control of important properties (such as the properties of powers). On the other hand, it enables students to get used to new expressions functional to other activities of the path, such as the justification of properties related to divisibility or, at a higher level, the formulation of conjectures (for example the identification of the law which expresses the number of the divisors of a given number).

The teacher involved in this discussion (T) is a novice in relation to the activities of the ArAl Project. When we shared with her the a priori analysis of the task, she displayed, unlike other novices, a good level of awareness of her own task and a clear idea of the objectives to be achieved during the discussion with students and of the knots to be disentangled. She thought that recognizing R and S as numbers would not represent a real difficulty for her students, because during the previous lessons they had encountered activities in which letters were used to represent numbers.[7]

Before providing the excerpt and our analysis, it is important to stress the philosophical position underlying the methodology of our work with teachers. Even if we carry out an a priori analysis of every activity with the teachers and focus on possible approaches that could foster a real understanding in pupils, we let the teachers freely choose how to develop the activities in their classes. In fact, we believe that giving them the possibility of highlighting, through the a posteriori joint analysis of the audio-recordings, the gap between what they have foreseen and what really happened in the classes represents a central element to promote their genuine professional development.

In line with this choice, we let T choose to change even the text of task: In the original version of the problem, in fact, the letters used to introduce the two numerical expressions (R and S) were not capital letters. The teacher decided to use capital letters because she thought it could be a better way to help students refer to them as names given to the expressions.

We chose to propose this excerpt because it highlights: (a) how typical arithmetical problems can be introduced through a pre-algebraic approach and how the use of letters foster the exploration of relationships, opening the way to generalization; (b) what kind of difficulties might be faced by the teacher and the students when early algebra is introduced in class (in particular, the effects of the teacher's unconscious choices); (c) the potential of the teacher's reflection after the activity to understand both the pupils' mental visions and their unpredicted interpretations, and aspects she might have neglected or interventions that might have been misleading for pupils; and (d) the effects of the exchange of comments with mentors.

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[7] The students had worked, for example, on these problems: (1) "What number is 'hidden' behind the letter a in this equality: $a + 12 = 20$?"; (2) "What numbers are 'hidden' behind the letters a and b in the equality $a + b = 18$?"

EXCERPT FROM A DISCUSSION[8]	COMMENTS GIVEN[9] DURING THE JOINT A POSTERIORI ANALYSIS
The teacher writes the following equalities on the blackboard, one above the other: $R = a \times 4 \times 9$ $S = 2^2 \times b \times 3^2$ 1. T: What can we say about R and S? 2. PPP: [silence]	1) T'S COMMENT: I thought that my pupils would have immediately understood that R and S were two different numbers and, most of all, that *a* and *b* could have any value in the given number field (natural numbers). But actually … silence! 1) M1'S COMMENT: Because of that silence, maybe it would have been better to decode those expressions, reminding them that the two capital letters were meant to give a "name" to the two numbers. 1) M2'S COMMENT: Why capital letters for R and S? Did you want them to focus on them? But without an explanation, it might have been misleading for the pupils
3. T: What is three to the two? 4. P1: Nine. 5. T: What is *a* in number R?	2) T'S COMMENT: Reflecting afterward, I'd say that I used the wrong strategy and question: In fact, my goal was to lead the pupils to "read" the same number represented in different ways, but also that they would get to argue about the possible link between R and S, drawing on their hypothetical thinking. 2) M2'S COMMENT: The question "What is *a* in number R?" is misleading. What did you expect? That they would say "A factor of R? A hidden number? Or rather a number I can choose as I like?" You could propose the problem of the comparison between R and S and recall the objective, saying, for instance, "R and S are two numerical expressions. I must decide whether they can represent the same number. How can I do that?" That might have been the first step to guide the

[8] In the discussion excerpt, the letter T introduces the teacher's interventions, and P introduces the pupils' interventions. The number that follows P indicates the pupil with that ranking number in the class list. PPP stands for several pupils who answer or intervene together.

[9] In this column, comments made by the teacher and by the two mentors (M1 and M2) are reported.

	pupils toward the comparison between the respective number factors. Did you think they would have immediately seen the equality between the factors 9 and 3^2, and also 4 and 2^2, and could directly argue about *a* and *b*?
6. P2, P1: Two to the two. 7. PPP: Two to the two. 8. T: Two to the two? This is weird. Is it because of the position with respect to S? 9. PPP: Yes. 10. T: How much is two to the two? 11. P3: Four. 12.T: In multiplication is the order in which I write the factors important? What have we always said for the commutative law? 13. PPP: Yes, … no.	3) T'S COMMENT: … "What have we always said … ?" I should have given a numerical example or rather invite my pupils to propose a simple situation of application of the commutative property in order to activate their skills and thus increase their confidence in the possibility of progressively solving the given task. 3) M2'S COMMENT: You might have written R as 4 × *a* × 9, asking them whether the new expression was still R or not, and then pose the problem again.
14. T: Why is *a* two to the two? Try and swap the factors; that is, the position of the numbers that are multiplied. Would the value of numbers R and S change? 15. PPP: [silence] 16. PPP: Yes … .	4) T'S COMMENT: This question was too difficult. … It confused my students, who were still looking for analogies between the factors of R and S and for the meaning of *a* and *b*. The initial question could have been repeated. I should have said: "What if I write the factors of R in another way?"
17. T: Sure? If I wrote R in another way and left S as it is written, would you still be stuck? (*I write R and S on the blackboard again*) 18. $R = 4 \times a \times 9$ 19. $S = 2^2 \times b \times 3^2$ 20. P2: *a* equals one, and *b* equals one!	5) T'S COMMENT: Pupil P2 realizes that there is an analogy in the structure and immediately says that *a* and *b* are equal to 1. This solves the problem about the value of R and S and opens the discussion to the other examples that give the same equality. Anyway, I could have asked what value R and S would have in that case.
21. T: What did you do? What value did you give *a* and *b*? 22. P2: The same value. 23. T: If *a* and *b* were	6) T'S COMMENT: The requests formulated later aim at generalizing the relation of equality between R and S whenever *a* and *b* have the same value. 6) M2'S COMMENT:

equal to 5? Would that change anything?	Here I would have recalled the pupil's reasoning and redirect the issue back to the class.
24. P2: No, because the result would always be the same. 25. T: Are R and S the same number or not?	7) T'S COMMENT: P2's answer puzzles me. Which result does he refer to? To the value of R and S when *a* and *b* are equal to 1? Or rather, by *result* does he mean the situation itself? Hence I tried to clarify it better. 2) M1'S COMMENT: I would have stimulated the class, asking: "Do you all agree?" or "Do you think I can make R and S different?"
26. P3: Yes, they are. 27. T: When? 28. P2 : When *a* and *b* have the same value. 29. T: Does the fact that *a* and *b* are in different positions confuse you? 30. PPP: [silence]	8) T'S COMMENT: Until now, the dialogue has been limited to a couple of students who do not always participate in classroom activities. The others keep silent, although they followed the bulk of the discussion. Therefore, I address the whole class to test how much the other students have followed. …
31. T: Can you change the position of the factors in a multiplication or not? 32. PPP: Yes … no. 33. T: Why don't you try to swap the factors? Any change in R and S? Make some trials, always leaving the same values for letter *a*. Do the same with *b*. Try and play with them. …	9) T'S COMMENT: Some pupils are still puzzled in tackling the task, so I write some of their proposals on the board and add my own.
META COMMENT BY THE TEACHER: The analysis of this class discussion was much more useful than the previous ones. I think that, by chance, the way I wrote the numbers—one above the other rather than one next to the other—created a situation that I didn't foresee in my prior analysis and didn't expect to happen. First, in previous situations the pupils had associated numbers with lowercase letters and not capital letters. This was a "detail" for me. I thought it would have been the same to use lowercase or caps, but that might have been misleading for the pupils; for instance, by thinking of R as a "set." I should or could have expected "spontaneous" argumentation about the nature of R and S, discussing possible values for *a* and *b*. Moreover, the initial question "What can we say about R and S" is too generic and open ended.	

Analysis of the excerpt

As both the first comments proposed by M1 and M2 highlight, from the very beginning of this discussion, the pupils faced a difficulty that T had underestimated in her planning of the activity: The intentional use of the capital letters R and S to introduce the two expressions produced a block in the pupils, preventing them from conceiving of these two expressions as numbers to be compared. Another unconscious choice made by T that contributed to creating this kind of block was related to the fact that on the blackboard she wrote the two expressions to be compared one above the other. This choice helped the pupils overlook the comparison between the two expressions and induced them to identify the values represented by the letters (Line 6), invalidating the enquiry the teacher had hypothesized concerning the meanings that they could get. Up to that point, the class had actively participated in the proposed activities, showing intelligence and awareness.

If we focus on the comments proposed by T, this excerpt stresses the teacher's attention to the problem of the formulation of the questions she had posed and her awareness about the importance of avoiding interventions that might be obscure or ambiguous for the pupils. Looking back, T was able to admit that the pupils' initial unpredicted confusion and the didactical "damages" might have been caused by some of her implicit assumptions and by occasional didactical choices (Comments 1 and 2).

Moreover, this MT also highlights the potential of this kind of a posteriori joint analysis of transcripts. In fact, T made a considerable number of high quality comments (increasing as the activity progressed). Her remarks also highlighted that she had become more autonomous regarding attitudes she should have in the classroom (Comment 3), questions she should pose (Comments 2, 4, and 5), and her own growing need to clarify didactical choices and objectives underlying some of her interventions (Comments 6, 7, and 8) by means of a dialogue with mentors and colleagues who would read the transcripts.

CONCLUDING REMARKS

This paper, in tune with the paradigm of the Italian research for innovation, is aimed at highlighting some significant wide-ranging effects of research results on teaching.

In the case analyzed, the interplay between theory and practice is evident. On the one hand, the paper shows how consolidated research results could be brought into lessons to foster a pre-algebraic approach to arithmetic. The problem on which the MT was focused exemplifies, in particular, the work we are conducting both to insert early-algebraic activities into the traditional curriculum and to involve reluctant teachers in working on innovative activities that are not usually part of the traditional mathematics curriculum.

On the other hand, the paper discusses specific methodological aspects concerning both how to deal with this kind of innovative activity in lessons and in teacher education. The example presented, in fact, shows potential problematic elements that have to be faced by the teacher during such activities and, at the same time, how the MTM allows teachers to objectify teaching and learning processes in order to evaluate their outcomes. This methodology also guides teachers through a reflection on the processes at different levels and from different points of view: that of the development of a mathematical construction by the pupils, that of the teacher's action, and that of the participation of the individuals in the collective construction of knowledge. Through this continuous work of analysis and reflection, the teacher realizes how the acquisition of competences and awareness by the pupils is strongly linked to the use of language, as well as to attitudes, actions, and choices of the teachers themselves. Through MTs, in fact, teachers get the chance to analyze the different interpretations pupils can give to the problems posed and how those interpretations can be strictly connected with the teacher's choices. MTs represent a chance for teachers to express their own difficulties and doubts, but also to make explicit their recently achieved awareness.

The teachers' route towards the acquisition of the ability of self-analysis, with a meta-attitude towards their own actions, and that of verbalization of their own reflections and awareness might be a slow and winding one. Similarly, it may be initially difficult to accept the idea that their MTs will be shared outside the small group: This is the moment when mentors play a crucial role, managing, most of the times, to help teachers understand that the methodology is meaningful and productive as far as they show to be ready to get involved in a peer, open, and sincere exchange (Malara & Navarra, to appear). Regarding this, we underline that the effectiveness of the MTM is based on two particular conditions. A first condition is that the activity of analysis and reflection is not an episodic one: It is only through continuous work that the teacher can really come to develop not only new awareness about algebra and its teaching, but also new attitudes and modalities of action. Another fundamental condition is that a relationship of mutual trust between teachers and researchers is constituted: This trust leads the teacher to develop a feeling of belonging to a group and also makes shared cultural and identity-related values become stable over time.

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